Ring Exchanges and the Supersolid Phase of $^4$He

D. M. Ceperley*
Department of Physics and NCSA, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA

B. Bernu
Laboratoire de Physique Théorique des Liquides, UMR 7600 of CNRS, Université P. et M. Curie, boîte 121, 4 Place Jussieu, 75252 Paris, France
(Received 15 September 2004; published 7 October 2004)

Using path integral Monte Carlo simulations we calculate exchange frequencies in bulk hcp $^4$He as atoms undergo ring exchange. We fit the frequencies to a lattice model and examine whether such atoms could become a supersolid, that is, have a nonclassical rotational inertia. We find that the scaling with respect to the number of exchanging atoms is such that superfluid behavior will not be observed in a perfect $^4$He crystal.

DOI: 10.1103/PhysRevLett.93.155303 PACS numbers: 67.80.–s, 02.70.Ss, 05.30.Jp

Recent torsional-oscillator observations by Kim and Chan on solid $^4$He, both in the disordered absorbed vycor [1] and in bulk $^4$He [2], have revived interest in the supersolid phase. In this phase, one has both long-range crystalline order and superfluidity. Experiments find a superfluid response to rotation at temperatures below 0.2 K. The low temperature superfluid density is about 1% of the solid density and is found to be independent of density within a pressure range of 25 to 60 bars. In the following, we assume the observed phenomena is described by equilibrium thermodynamics of pure bulk helium. In practice, whether this is the case is one of the most important issues. Assuming equilibrium [3], a supersolid is characterized by both long-range translational order and by a nonclassical response to rotation (NCRI).

In proposing such a state, Andreev and Lifshitz [4] postulated the existence of suitable defects (e.g., vacancies) and then estimated the properties of the dilute system of bosonic defects. The superfluid transition temperature of point bosons is $T_c = 3.31 \rho^{2/3} \hbar^2/m$ where $\rho$ is the density of the Bose condensing quasiparticles and $m$ their effective mass. Taking the measured value for $T_c = 0.2$ K, writing the mass in terms of the bare helium atom mass $m = \mu m_d$ and their density as a fraction $c$ of the bulk helium density at melting $\rho = 0.029$ Å$^{-3}$ we find that $c = 0.012 \mu^{3/2}$: if the quasiparticle is heavier than a helium atom, at least one out of every 83 lattice sites must have a defect; if the mass is 1/10 the atom mass, the relative density of defects will need to be $4 \times 10^{-4}$. Such a dense collection of defects could not be associated with the much lower concentration of $^3$He impurities.

Experiments, particularly the NMR experiments [5,6] on solid $^3$He rule out zero point vacancy concentrations of more than one part in $10^{14}$. Pederiva et al. [7] have done calculations of the energy of vacancies and find an energy of about $15 \pm 4$ K in the hcp phase of bulk $^4$He at the melting density, and $30$ K at $50$ bars, in agreement with various experiments, most notably x-ray scattering [8].

Using path integral calculations (described below) we estimate the energy to create an interstitial at the melting density is $48 \pm 5$ K; they are even more improbable than vacancies. Hence, the energies of point defects are substantially greater than zero; in equilibrium there should not be enough of them at 0.2 K to Bose condense; this precludes the Andreev mechanism for a supersolid.

In this Letter we examine the possibility that bulk solid hcp helium, assumed to be free of defects such as impurities and vacancies, could have a supersolid phase. One might think that there would always be ground state defects, arising from the large quantum zero point fluctuations. Near melting, the rms vibration about the lattice site is 30%, so that at any instant of time, a good fraction of atoms are closer to a neighboring site than to their home site. However, the absence of an atom from a lattice site is not sufficient for having a supersolid; if the empty site is always accompanied by a doubly occupied site, there can be no mass current. Chester [9] proved that any Jastrow (i.e., pair product) wave function of finite range has both Bose-Einstein condensation (BEC) and vacancies. However, Jastrow wave functions crystallize with much difficulty and with a transition density off by an order of magnitude [10] so that they are not reliable enough to be used to predict superfluidity. In general, variational wave function approaches are suspect, since the energy is not sensitive to the low probability regions of configuration space that are important for the superfluid density and Bose condensation.

Leggett [3] has written an upper bound to the superfluid density in terms of the zero temperature density of the solid. Using densities from path integral calculations, we find that the Leggett bound gives $\rho_s/\rho \leq 0.16$ at the melting density, a value consistent with experiment. In this Letter, we calculate values for ring exchange fre-
quencies and using them, find that the superfluid density should be zero in solid helium.

To determine whether bulk helium could be supersolid, path integrals give a much cleaner framework; they can be used to compute the superfluid density and the momentum distribution without the assumption of a trial wave function or any other uncontrolled approximation. The partition function of $N$ bosons is given by

$$Z = \frac{1}{N!} \sum_{PF} \int dR \langle e^{-\beta H} | PR \rangle,$$

where $H$ is the Hamiltonian, $\beta$ the inverse temperature, and $R = \{r_1, r_2, \ldots, r_N\}$. For numerical calculations, the density matrix operator is expanded into a path, beginning at the configuration $R$ and ending at $PR$. In terms of these paths, the superfluid density (i.e., the number of atoms not moving with the walls of the torsional-oscillator) is given by

$$\rho_s = \frac{m \langle \hat{W} \rangle}{\hbar^2 B N},$$

where $\hat{W} = \int_0^\beta dt \sum_{i=1}^N \{\mathbf{r}_i(t)\}/dt$ is the winding number of the path around a torus. It is only exchanges on the order of the sample size that contribute to the superfluid density; local exchanges make no contribution. Using the path integral Monte Carlo (PIMC) method, superfluidity and freezing happen naturally at the right density and temperature, without imposing them in any way. The technical complications [11] concern ergodicity of the random walk, and finite size effects: one has to take the limit as $N \to \infty$.

PIMC calculations find a superfluid density on the order of $3\%$ at melting density (molar volume 21.04 cm$^3$) and about $1.2\%$ at 55 bars (molar volume 19.01 cm$^3$) in a 48 atom ($3 \times 4 \times 2$) hcp supercell. The superfluid density is larger than what is observed, and has a relatively weak pressure dependence. However, a cell with 180 atoms, has zero superfluid density. We cannot be sure that the lack of winding paths has a physical origin, or is due to a lack of ergodicity within the PIMC random walk. To change from one winding configuration to another involves a global move of the paths which becomes very unlikely as the box size gets large. To avoid this problem, we turn to a PIMC approach which directly estimates individual exchange probabilities.

The Thouless [12] theory of exchange in quantum crystals assumes that at low temperatures, the system will almost always be near one of the particular $N!$ arrangements of particles to lattice sites with rare, rapid, tunnelings from one arrangement to another. Hence, we label the particles with their initial lattice sites. Then the partition function is written as a sum over permutations of sites onto themselves. We break up the permutation into cyclic exchanges $\{p_1, p_2, \ldots, p_n\}$; each cycle is independent, so that

$$Z = Z_0 \prod_{j=1}^{n_p} f_{p_j}(\beta),$$

where $Z_0 = \int dR \langle e^{-\beta H} | R \rangle$ is an uninteresting phonon partition function at low temperature. The contribution for a cycle

$$f_{p_j}(\beta) = \frac{1}{Z_0} \int dR \langle e^{-\beta H} | pR \rangle = J_{p_j} \beta$$

is proportional to $\beta$ because it is localized in time; the coefficient, $J_{p_j}$ is the exchange frequency.

A special case of the above partition function is the Feynman-Kikuchi (FK) model [13] that assumes that $J_{p_j} = J_0 e^{-a L_{p_j}}$ where $L_{p_j}$ is the cycle length. The partition function is

$$Z = Z_0 \sum_P (\beta J_0)^{n_p} e^{-a} \sum_{\ell_1, L_{p_1}} \cdots \sum_{\ell_n, L_{p_n}}.$$
exponentially with the length of the exchange with an exponent of $\alpha = 2.64$ near the melting volume $21.04 \text{ cm}^3$, and $\alpha = 3.14$ at the volume $19.01 \text{ cm}^3$ corresponding to $P = 60$ bars.

To construct a more realistic model than the FK model, we need to take into account more details of the geometry of the exchange than just the number of exchanging atoms. We assume that it is the internal geometry of the exchange that matters [18]; the detailed arrangement of the neighboring spectator atoms is much less important. In particular, we assume the action of a given $n$ cycle (the log of the exchange frequency) is the sum over the internal vertices of the exchange

$$J_p = J_0 \exp \left[ - \sum_{k=1}^{n} \alpha(\theta_k) \right].$$

(6)

Here $\theta_k$ are angles between successive displacements in the exchange cycle. In an hcp lattice there are seven possible angles between two nearest neighbor displacement vectors, but $\theta = 0$ occurs only in the pair exchange. We determine the parameters by fitting to the PIMC exchange frequencies $J_p$. Good fits are obtained; the model predicts the exchanges frequencies with an accuracy of about 20%. The resulting coefficients are shown in Fig. 2. Typical errors on the coefficients are $\pm 0.03$. Note that there is a strong preference for exchanges that proceed in a straight line versus ones which have sharp angles; for them the incoming and outgoing particles are more likely to collide. The fitting coefficients are seen to depend linearly on $\cos^4(\theta/2)$. We find no preference for exchange in the basal plane beyond the effect induced by the selection of angles.

We also verified this model by comparing with results of the direct PIMC calculations (where permutations are generated dynamically) mentioned above and obtained rough agreement with the winding numbers and permutation cycle distributions. The two and three-particle exchanges are about twice the prediction of model. On the other hand, the exchanges in the 180 atom cell are somewhat smaller than Thouless theory. We expect such disagreements because Eq. (6) does not contain the effect of spectator atoms on the exchange process, and small exchanges have not been included in the fit.

The activation coefficient of the exchange is $J_0 = 7.2 \pm 0.5 \text{ K}$. This is the order of magnitude but smaller than the vacancy energy. If it is related to vacancies, then $J_0$ should be of the same order of the vacancy energy at all

![Graph](image-url)
densities, but on the contrary, we do not find a significant density dependence.

We now analyze the model. Because the probability of retracing is small, we initially neglect the self-intersections. This leads to a diffusion problem on a lattice, with a large probability of continuing in the same direction. Since a displacement only depends on the previous displacement, it is an un-normalized Markov process. Because the hcp lattice has a basis, we have to label the displacements consistently on the A and B planes so that the transition probabilities are independent of the plane.) The probability of a given displacement vector \( \Pi_i \) will approach a steady state after many steps, \( \Pi_i e^{\lambda \tau} = \lambda \Pi_j \).

By symmetry \( \Pi_i \) can only depend on whether the direction is in the basal plane \( \Pi_1 \) or out of the basal plane \( \Pi_2 \). Define the partial sum \( D_{ij} = \sum_i e^{-\alpha \tau} \) where \( i, j \) are either in the basal plane (1) or out of it (2), e.g., \( D_{11} \) is the sum of the probability of both vectors being in the basal plane). The eigenvalue \( \lambda \) is the solution to the secular equation

\[
(D_{11} - \lambda)(D_{22} - \lambda) - D_{12}^2 = 0.
\]

Putting in the model parameters, we find \( \lambda = 0.303 \pm 0.005 \). The probability of having an exchange of length \( \rho \) will equal a prefactor times \( \lambda^\rho \), hence, since \( \lambda < 1 \), our PIMC results imply that solid \(^4\)He will have only localized exchange and thus cannot be a supersolid.

Let us consider how the neglect of the self-intersections affects the critical value. Qualitatively, self-intersections must decrease the probability of long exchange cycles. To estimate the effect quantitatively, we perform random walks on the hcp lattice and count only nonintersecting walks. If we artificially change the model parameters \( \alpha \) by subtracting 1.2, the model becomes critical when \( \lambda = 1.03 \) as opposed to \( \lambda = 1 \). This line is shown in Fig. 2. Our computed value of \( \lambda \) is much less than what is needed to allow for a supersolid. We expect that the presence of other exchange cycles will further increase the critical value of \( \lambda \).

In summary, PIMC-computed exchange frequencies for hcp solid \(^4\)He show that only localized exchanges will be present and thus should not exhibit the property of nonclassical rotational inertia. Based on other theoretical and experimental findings, we think it unlikely that the observed phenomena are due to vacancies, interstitials or \(^3\)He impurities. Hence, one must look for an explanation of the experiments elsewhere, either to nonequilibrium effects or more complicated lattice defects.

This research was supported by the NASA program in Fundamental Physics and the Department of Physics at the University of Illinois Urbana-Champaign.

\*Electronic address: ceperley@uiuc.edu