Momentum Distribution of the Homogeneous Electron Gas

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We calculate the off-diagonal density matrix of the homogeneous electron gas at zero temperature using unbiased reptation Monte Carlo calculations for various densities and extrapolate the momentum distribution and the kinetic and potential energies to the thermodynamic limit. Our results on the renormalization factor allow us to validate approximate $G_0W_0$ calculations concerning quasiparticle properties over a broad density region ($1 \leq r_s \leq 10$) and show that, near the Fermi surface, vertex corrections and self-consistency aspects almost cancel each other out.

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The uniform electron gas (jellium) is one of the most fundamental models for understanding electronic properties in simple metals and semiconductors. Knowledge of its ground state properties and, in particular, of modifications due to electron correlation is at the heart of all approximate approaches to the many-electron problem in realistic models. Quantum Monte Carlo methods (QMC) [1] have provided the most precise estimates of the correlation energy, electron pair density, and structure factor of jellium, basic quantities for constructing and parametrizing the exchange-correlation energy used in density functional theory [2].

Correlations modify the momentum distribution, $n_k$, of electrons, and introduce deviations from the ideal Fermi-Dirac step function. The magnitude of the discontinuity at the Fermi surface ($k_F$), the renormalization factor $Z$, quantifies the strength of a quasiparticle excitation [3] and plays a fundamental role in Fermi liquid and many-body perturbation theory (GW) for spectral quantities. Whereas the momentum distribution (as well as other spectral information) is inaccessible in current Kohn-Sham density functional theory formulations, the reduced single-particle density matrix—the Fourier transform of $n_k$ of jellium, basic quantities for constructing and parametrizing the exchange-correlation energy used in density functional theory [4].

In this Letter, we calculate $n_k$ for the electron gas (jellium) by QMC calculations in the density region $1 \leq r_s \leq 10$. Here, $r_s = (4\pi n a_B^3/3)^{-3/2}$ is the Wigner-Seitz density parameter, $n$ is the density, and $a_B = h^2/m_e^2$ is the Bohr radius. In contrast to previous calculations [6], our calculations are based on more precise backflow (BF) wave functions [7], and a careful extrapolation to the thermodynamic limit [8,9]. Similar to the worm algorithm in finite temperature path-integral and lattice Monte Carlo [10,11] calculations, we have extended reptation Monte Carlo (RMC) calculations[12] to include the off-diagonal density matrix in order to obtain an unbiased estimator of the momentum distribution [13,14]. From our extrapolation scheme, we derive the exact behavior of $n_k$ close to the Fermi surface. By comparing the renormalization factor, $Z$, with different approximate GW theories, we can judge the importance of self-consistency and vertex corrections within these approaches. The excellent agreement of our QMC results with $G_0W_0$ over a broad density region indicates strong cancellations of vertex and self-consistency corrections close to the Fermi surface.

Within variational Monte Carlo (VMC) calculations, the ground state wave function is approximated by a trial wave function, $\Psi_T(\mathbf{R})$, whereas within projector Monte Carlo methods, e.g., diffusion Monte Carlo (DMC) or RMC calculations, the trial state is improved using $\Psi_P \propto \exp[-\beta H] \Psi_T$; this converges exponentially fast to the true ground state for increasing projection time $\beta$. To circumvent the so-called fermion sign problem, calculations are done within the fixed-node approximation, introducing small systematic deviations from the exact fermion ground state [15]. Whenever the (approximate) nodes of the system are described by a determinant of single-particle orbitals $\phi_n(\mathbf{r})$, the (fixed-node) ground state wave function, $\Psi_N(\mathbf{R})$, of $N$ particles at positions $\mathbf{R} = \{\mathbf{r}_i\}$ can be written as

$$\Psi_N(\mathbf{R}) = D_N \exp[-U_N], \quad D_N = \det{\phi_n(\mathbf{r}_i + \nabla W_N)}.$$

(1)
where \( W_N \) and \( U_N \) are generalized backflow and Jastrow potentials [16], respectively.

From an approximate ground state wave function, \( \Psi_N(R) \), we obtain the reduced single-particle density matrix [17]

\[
\Psi_N(R) = \langle F(R; r) \rangle_N, \quad F = \frac{1}{N} \sum_i \Psi_N(R, r_i + r) \rangle_{\Psi_N(R)}, \tag{2}
\]

where \( R, r_i + r \) indicates that the position of particle \( i \) is displaced by \( r \). To verify that \( P \) and \( Q \) as defined by Eq. (4) respectively.

The Fourier transform of \( f_N(r) \) directly yields the momentum distribution, \( n^N_k \), of the electrons per spin

\[
n^N_k = \frac{1}{2V} \int \cos(k \cdot r) f_N(r), \tag{3}
\]

where \( V \) is the volume.

The large variance of the estimator of the off-diagonal density matrix, Eq. (2), makes precise calculations very time-consuming. To reduce the variance for homogeneous systems with plane wave orbitals, \( \phi_n(r) = e^{ik_n \cdot r} \), we separate the ideal gas structure factor, \( n^N_k \), into the momentum distribution, \( n^N_k \), and the momentum distribution, \( n^N_k \).

Quantum Monte Carlo results are obtained for typically \( N \approx 10^3 \) electrons. The extrapolation to the thermodynamic limit introduces important quantitative and qualitative changes of the momentum distribution around the Fermi surface, \( k_F \) [9]. For a homogeneous periodic system, the orbitals are plane waves, \( \phi_n(r) = \exp[i \cdot (k_n + \theta) \cdot r] \), in the Slater determinant of Eq. (1), where \( k_j \in G_N = \{(n_1, n_2, n_3)2 \pi V^{-1/3} \} \) with integer \( n_k \), and \( \theta \) can be chosen to introduce twisted boundary conditions [8,20]. For a normal Fermi liquid, we further have \( |k_j + \theta| \leq k_F \), and the generalized backflow and Jastrow potential \( W_N \) and \( U_N \) can be written exclusively in terms of collective coordinates \( \rho_k = \sum_n \exp[k_n \cdot r] \), and their gradients [7,16]. Using the wave function "potentials," \( W_N \) and \( U_N \), expressed as continuous functions in terms of the collective coordinates, the relation between the wave function in the limit \( N \rightarrow \infty \) to a finite system is well defined, as it just amounts to evaluations on a denser grid in \( k \) space [8,9].

Let us first discuss the finite size scaling for a Slater-Jastrow (SJ) wave function: a determinant with \( W_N \equiv 0 \), together with a two-body Jastrow correlation, \( U_N = \sum_{1 \leq k \leq 1/2} u_k \rho_k \cdot \rho_{k+1/2} \). We further assume that the function \( u_k \) is analytically given. In their SJ-VMC calculations, we use the Gaskell form \( u_k = S_0^{-1}(k) + S_0^{-2}(k) + 2n v_k / e_k \) where \( S_0(k) \) is the ideal gas structure factor, \( v_k = 4 \pi e^2 / k^2 \), and \( e_k = h^2 k^2 / 2m \) [21,22]. Neglecting mode coupling between single-particle modes in \( D_N \) and collective modes described by \( U_N \), the single-particle density matrix, Eq. (2), can be approximated as

\[
\frac{D_N}{D_N} \langle e^{-(U_N - U_N)} \rangle_N, \tag{6}
\]

where the prime indicates the off-diagonal configuration, e.g., \( D_N \equiv D_N(R, r_1 + r) \). Within the cumulant and rotating wave approximation, we then obtain an explicit expression,

\[
f_s(r) \approx f_{id}(r) \exp[-s(N)], \tag{7}
\]
TABLE I. The total \((E), \text{ potential } (V), \text{ and kinetic energy } (T) \) per particle in Ry, and the contact value of the pair correlation function \(g(0), \) all extrapolated to the thermodynamic limit from unbiased RMC calculations with BF nodes. We further give parameters of the momentum distribution at small \(k \) \((n_0, \text{ and } n_2), \) \(n(k \rightarrow 0) = n_0 - n_2(k/k_F)^2, \) and at \(k_F, \bar{n} = \{n(k_F) + n(k_F)\}/2. \)

<table>
<thead>
<tr>
<th>(r_s)</th>
<th>1</th>
<th>2</th>
<th>3.99</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E)</td>
<td>1.173(2)</td>
<td>0.0039(1)</td>
<td>(-0.155(1))</td>
<td>(-0.1520(1))</td>
<td>(-0.1071(1))</td>
</tr>
<tr>
<td>(T)</td>
<td>2.290(3)</td>
<td>0.6024(5)</td>
<td>0.1688(1)</td>
<td>0.1131(1)</td>
<td>0.0349(1)</td>
</tr>
<tr>
<td>(V)</td>
<td>(-1.116(1))</td>
<td>(-0.5985(1))</td>
<td>(-0.3243(1))</td>
<td>(-0.2651(1))</td>
<td>(-0.1421(1))</td>
</tr>
<tr>
<td>(g(0))</td>
<td>0.268(3)</td>
<td>0.152(2)</td>
<td>0.057(2)</td>
<td>0.034(1)</td>
<td>0.0036(4)</td>
</tr>
<tr>
<td>(n_0)</td>
<td>0.999</td>
<td>0.998</td>
<td>0.97</td>
<td>0.93</td>
<td>0.88</td>
</tr>
<tr>
<td>(n_2)</td>
<td>0.038</td>
<td>0.066</td>
<td>0.12</td>
<td>0.098</td>
<td>0.21</td>
</tr>
<tr>
<td>(\bar{n})</td>
<td>0.490</td>
<td>0.477</td>
<td>0.460</td>
<td>0.456</td>
<td>0.414</td>
</tr>
</tbody>
</table>

\[
x_N(r) = \frac{1}{V} \sum_{|k| \leq k_c} [u_{k}(S_{k} - 1) + nu_{k}^{2}S_{k}] e^{ik\cdot r} - 1, \quad (8)
\]

where \(S_{k} = \langle \rho_{k}\rho_{-k}\rangle_{N}/N \) is the structure factor, \(f_{id}(r) = 2\sum_{k \leq k_c} e^{ik\cdot r}/N \) is the single-particle density matrix of the corresponding ideal gas, and we have neglected contributions of short wavelength modes, \(k_c = 0.48 r_F^{-1/2} k_F \). Further, we can use \(S_{k} = [2nu_{k} + 1/S_{0}(k)]^{-1} \) to express \(S_{k} \) in terms of \(u_{k} \) and \(S_{0}(k) \) which is based on assuming Gaussian statistics for \(\rho_{k} \) so that Eq. (7) gives an explicit expression for \(f_{N}(r) = f_{c}(r) \) in terms of a given Jastrow factor. Whereas the resulting model, Eq. (7), depends weakly on \(k_c \), so that \(f_{N}(r) \) and \(n_{k} \) are only qualitatively described, the size extrapolation is quantitatively correct, as it is dominated by the Jastrow singularity \(u_{k} \rightarrow (v_{k}/2n_{p})^{1/2} \) and \(S_{k} \rightarrow (2nv_{k}/v_{0})^{-1/2} \) for \(k \rightarrow 0 \) stemming from the plasmon contributions.

Since we expect that mode coupling is negligible in the long wavelength limit, the cumulant expression, Eq. (7), can be used to determine the size corrections of QMC calculations of the finite system

\[
f_{\infty}(r) = \frac{2}{n} \int \frac{d^{3}k}{(2\pi)^{3}} n_{k}^{N} e^{ik\cdot r} e^{-(x_{0}(r) - x_{N}(r))}. \quad (9)
\]

Here \(n_{k}^{N} \) is the momentum distribution of the \(N \) electron system, defined for all values of \(k \) in a grand canonical ensemble using twisted boundary conditions [8]. Using the long-range properties of \(u_{k} \) and \(S_{k}, x_{0}(r) \) is obtained from Eq. (8) in the limit \(N \rightarrow \infty \). From the Fourier transform of \(f_{\infty}(r), \) Eq. (9), we obtain the extrapolated momentum distribution, \(n_{k}^{\infty} \). A related linearized expression has been used to extrapolate \(n_{k} \) of the two-dimensional electron gas using BF-VMC [9] calculations.

Following the analysis of Ref. [9], leading order corrections to the renormalization factor, \(Z_{N} = n_{k_{F}}^{\infty} - n_{k_{F}}^{N} \), are given by

\[
Z_{\infty} \approx Z_{N} \exp[-\Delta_{N}]
\]

\[
\Delta_{N} = \int_{-\pi/L}^{\pi/L} \frac{d^{3}q}{(2\pi)^{3}} \frac{u_{q}}{2} [1 + O(2nu_{q}S_{0}(q)]^{-1} \]

\[
= c \left( \frac{3}{4\pi} \right)^{1/3} \left( \frac{r_{F}}{3} \right)^{1/2} N^{-1/3} + O(N^{-2/3}), \quad (10)
\]

where \(c \approx 1.221 \) is a numerical factor to account for the cubic integration volume [24]. Whereas the asymptotic region is only reached for large systems with \(N^{1/3}r_{F}^{1/2} \gg 1 \), the extrapolation based on the full expression, Eq. (9), includes corrections beyond the leading order term. Analyzing Eq. (9) around \(k_{F} \), we obtain the exact leading order behavior with an infinite slope at \(k_{F} \):

\[
n(k \rightarrow k_{F}^{\pm}) \approx n(k_{F}^{\pm})
\]

\[
+ \frac{Z_{\infty}}{2\pi} \left( \frac{9}{4} \right)^{1/3} \left( \frac{r_{F}}{3} \right)^{1/2} \log \left| \frac{k}{k_{F}} - 1 \right|.
\]

FIG. 1 (color online). The momentum distribution \(n_{k} \) of the unpolarized electron gas for various densities extrapolated to the thermodynamic limit. The inset shows the extrapolation of \(n_{k} \) for \(r_{s} = 5 \) from a system with \(N = 54 \) electrons to the thermodynamic limit, \(N \rightarrow \infty \), leading to a significant reduction of the renormalization factor \(Z \).
Size extrapolation, discussed above, requires the knowledge of the structure factor, $S_k$, and the Jastrow potential, $u_k$, in Eq. (8). The QMC calculation of the $N$-particle system allows us only to determine them on a finite grid in $k$ space, but the analytic continuation to the dense grid can be done by interpolation from their known behavior at small $k$ [8]. Whereas $S_k$ can be calculated directly, $u_k = u_k^{SJ}$ is only known explicitly for VMC calculations using a Slater-Jastrow trial function. In general, imaginary time projection and backflow introduce an effective Jastrow potential, $u_k$, different from the explicitly given form of the underlying trial wave function. Expecting small changes at long wavelength, $u_k = u_k^{SJ} + \delta u_k$, we obtain the modifications $\delta u_k$ from changes in the structure factor $\delta S_k = S_k - S_k^{SJ}$ by linear response. For our purpose, mode coupling can be neglected, as well as deviations from Gaussian statistics, so that $\delta S_k / \delta u_k = -2nS_{-k}^{-2}\delta_{k,k'}$ for $k \to 0$. Modifications due to $\delta u_k$ have been used to estimate the systematic error of the finite size extrapolation.

Using SJ-VMC calculations with $u_k^{SJ}$ for $N = 54$ to $N = 1024$ electrons, we have checked that size extrapolations based on Eq. (9) with $N = 54$ are reliable. We have further checked for larger systems ($N = 342$) that our backflow wave functions and the imaginary time projection do not modify the long-range behavior already present in SJ-VMC calculations, so that the analysis above can be also applied to BF-VMC and RMC calculations. Thus, the more expensive backflow VMC and RMC calculations based on the analytical wave functions in Ref. [7] are only done with $N = 54$. Extrapolated results on the total energy $E$, unbiased estimators from reptation for the potential $V$ and kinetic energies $T$, and the contact value of the pair correlation function, $g(0)$, are given in Table I. The momentum distribution is shown in Fig. 1. The values for the renormalization factor, $Z$, together with different perturbative results from the literature are given in Table II. Note that the BF-VMC value $Z = 0.66(2)$ at $r_s = 3.99$ of Ref. [5] was based on the extrapolation of systems containing up to $N = 342$ electrons. This is an explicit check on the extrapolation procedure. Table I also contains the values of the momentum distribution at the origin, $n_0$, the negative slope at the origin, $n_2$, and $\bar{n} = (n_{k^+} + n_{k^-})/2$. These values can be used to parametrize the momentum distribution along the lines given in Ref. [29], together with $Z$, the exact large $k$ asymptotics [30], $n(k \to \infty) = (9/2)|r_s|^2g(0)/k^8$, and the exact behavior close to the Fermi surface, Eq. (11). Whereas the mixed estimator usually employed in DMC calculations introduces a small bias in the momentum distribution, size extrapolation introduces large systematic modifications which limit the precision of the calculations. Previous DMC results [6], using mixed estimators and SJ nodes, suffer from these strong finite size effects and overestimate $Z$ by a large amount.

In summary, we have calculated the momentum distribution using a new unbiased and much more accurate Monte Carlo method, and extrapolated the results to the thermodynamic limit. In particular, our data allow a quantitative comparison of the renormalization factor, $Z$, with approximate calculations (see Table II). The excellent agreement of our results with $G_0W_0 [25,31,32]$ over the whole metallic density region $r_s \leq 5$ strongly indicates that vertex corrections and self-consistency issues – neither is included in $G_0W_0$—are canceling each other, at least close to the Fermi surface.

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This variance reduction can be generalized to periodic systems with Bloch orbitals, 
\( \phi_{nk}(r) \sim e^{iK \cdot r} \varphi_n(r) \), displacing only the phase in the determinant, 
\( \phi_{nk}(r) \rightarrow e^{iK \cdot r} \phi_{nk}(r) \), when 
\( r_i \rightarrow r_i + r \).