BELL'S THEOREM, ENTANGLEMENT, TELEPORTATION, QUANTUM CRYPTOGRAPHY, QUANTUM COMPUTING AND ALL THAT

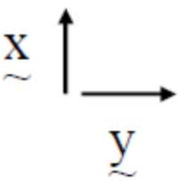
A. J. Leggett
University of Illinois at
Urbana-Champaign

Support: NSF E1A-01-21568

1. Elementary considerations on classical EM radiation and photons.
2. Bell's theorem
3. The general notion of entanglement
4. Some applications
5. Superconducting qubits – recent advances

I. Light waves and photons

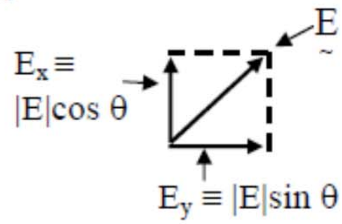
Classical light wave ($\underset{\sim}{k}$ into screen)



$$\underset{\sim}{E} = (\mathrm{Re})\ \mathrm{const.}\ \underset{\sim}{\hat{x}}\ \exp i(kz - \omega t)$$

$$\underset{\sim}{E} = (\mathrm{Re})\ \mathrm{const.}\ \underset{\sim}{\hat{y}}\ \exp i(kz - \omega t)$$

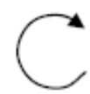
Principle of (classical) superposition:



$$\underset{\sim}{E} = (\mathrm{Re})\ (E_x\ \underset{\sim}{\hat{x}} + E_y\ \underset{\sim}{\hat{y}})\ \exp i(kz - \omega t)$$ also solution

note: E_x, E_y may be complex, e.g.

$$\underset{\sim}{E} = \mathrm{const.}\ \ (\mathrm{Re})\left\{(\underset{\sim}{\hat{x}} + i\ \underset{\sim}{\hat{y}})\ \exp i(kz - \omega t)\right\}$$

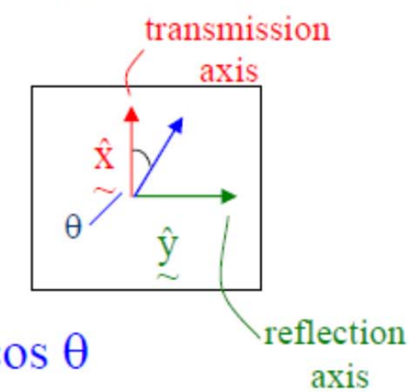


$$(\equiv \mathrm{const.}\ \underset{\sim}{\hat{x}}\ \cos(kz - \omega t) - \underset{\sim}{\hat{y}}\ \sin(kz - \omega t))$$

carries finite angular momentum.

- - - - - - - - - - - - - - - - - - - -

BIREFRINGENT CRYSTAL
("polarizer"):



Transmitted amplitude = $E_x = |E| \cos\theta$

Transmitted energy $\propto E_x^2 = |E|^2 \cos^2\theta$

(initial energy $\propto |E|^2$)

$\Rightarrow$ fraction of energy transmitted = $\cos^2\theta$

("Malus's law")

QM: Photons

QM amplitude $|\psi\rangle \Longleftrightarrow$ classical field amplitude $\underset{\sim}{E}$:

in particular, if $|\psi_1\rangle$, $|\psi_2\rangle$ allowed, so is superposition

$$\alpha|\psi_1\rangle + \beta|\psi_2\rangle$$

e.g. if

$|\psi_1\rangle \equiv |\underset{\sim}{\hat{x}}\rangle$ $\uparrow$ $\qquad$ $|\psi_1\rangle \equiv |\underset{\sim}{\hat{y}}\rangle$ $\longrightarrow$

single photon polarized along x-axis $\qquad$ single photon polarized along y-axis

then

$$|\psi\rangle = \cos\theta\, |\underset{\sim}{\hat{x}}\rangle + \sin\theta\, |\underset{\sim}{\hat{y}}\rangle$$

describes single photon with (linear) polarization at angle θ in xy-plane

and

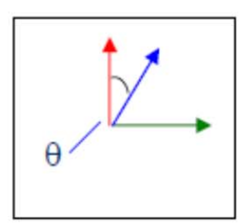
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\underset{\sim}{\hat{x}}\rangle \pm i\, |\underset{\sim}{\hat{y}}\rangle\right)$$

describes photon with R(L) circular polarization

(angular momentum $\pm\hbar$)

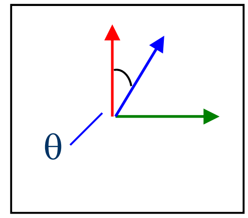
\- -

Single photon incident on birefringent crystal (polarizer):



Probability of transmission = $\cos^2\theta$

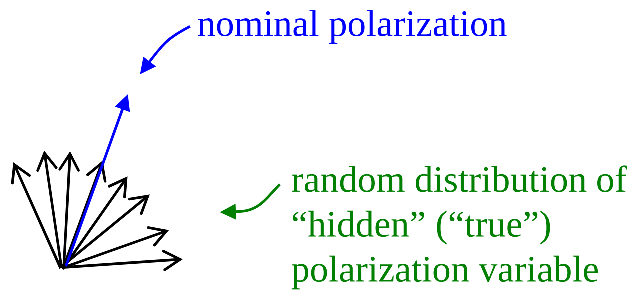
Single photon incident on
birefringent crystal (“polarizer”):



Probability of transmission = cos2q

(quantum version of Malus’ law)

Digression: Can a classical probabilistic theory explain this?



YES!

“true” polarization

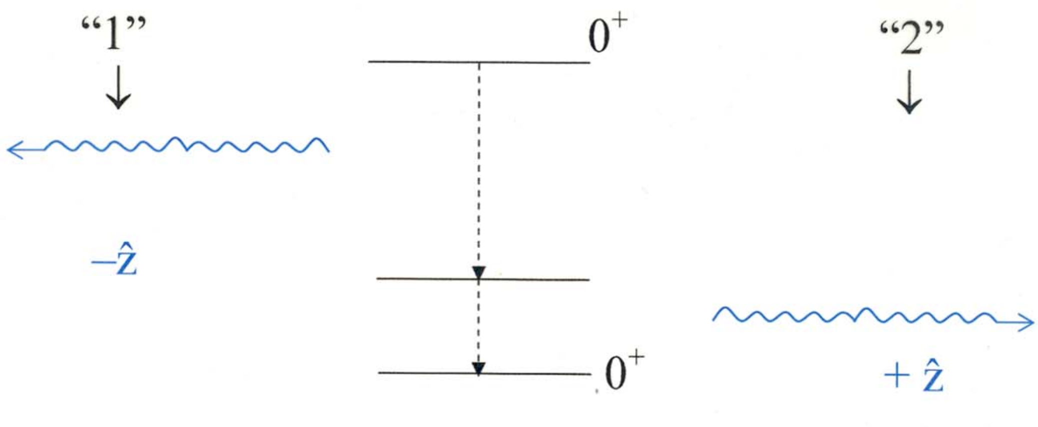
If $\nearrow$ closer to $\uparrow$, transmitted: if closer to $\longrightarrow$, reflected.
If transmitted, distribution of “hidden” variable is adjusted:

With suitable choice of random distribution, can

reproduce $P_T(\theta) = \cos^2 \theta$

$f(\chi) = \cos 2\,(\theta - \chi)$

2. 2-PHOTON STATES FROM CASCADE DECAY OF ATOM



ATOM

What is polarization state of photons?

(note: $|x\rangle \pm i|\hat{y}\rangle$ corr. photon angular momentum $\pm\hbar$)

General principle of QM; if process can happen either of two ways, and we don't (can't) know which, must add amplitudes!

Here, we know that total angular momentum of 2 photons is zero, but we don't know whether photon 1 carried off $+\hbar$ and 2 $-\hbar$ (intermediate atomic state has m = 1) or vice versa (int. state m = -1). Hence must write

crucial!

$$|\psi\rangle = |x + iy\rangle_1\,|x - iy\rangle_2 + (\text{phase factor}).\,|x - iy\rangle_1\,|x + iy\rangle_2$$

Actually (parity $\Rightarrow$) phase factor = + 1, so

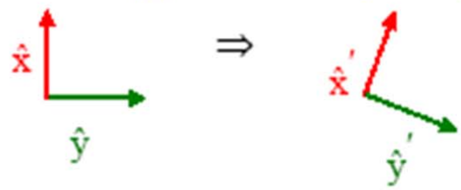
$$|\psi\rangle = |x\rangle_1\,|x\rangle_2 + |y\rangle_1\,|y\rangle_2 \quad \left(\times\frac{1}{\sqrt{2}}\right)$$

POLARIZATION STATE OF 2 PHOTONS EMITTED BACK TO BACK IN ATOMIC $0^+ \rightarrow 1^- \rightarrow 0^+$ TRANSITION (recap):

$$\Psi_{2\gamma} = \frac{1}{\sqrt{2}} \left(|x\rangle_1 |x\rangle_2 + |y\rangle_1 |y\rangle_2\right)$$

So, if photon 1 is measured to have polarization x(y) so inevitably will photon 2!

But, state is rotationally invariant:



$$\Psi_{2\gamma} = \frac{1}{\sqrt{2}} \left(|x'\rangle_1 |x'\rangle_2 \;+\; |y'\rangle_1 |y'\rangle_2\right)$$

so if 1 measured to have polarization x' (y') so does 2!

(NOT true for "mixture" of $|x\rangle_1 |x\rangle_2$ and $|y\rangle_1 |y\rangle_2$)

Now:

What if photon 1 is incident on polarizer with "transmission" axis $\underset{\sim}{\hat{a}}$. and photon 2 on one with a <u>differently</u> oriented transformation axis $\underset{\sim}{\hat{b}}$?

Since choice of axes for $\Psi_{2\gamma}$ arbitrary, choose $\underset{\sim}{\hat{x}} = \underset{\sim}{\hat{a}}$.

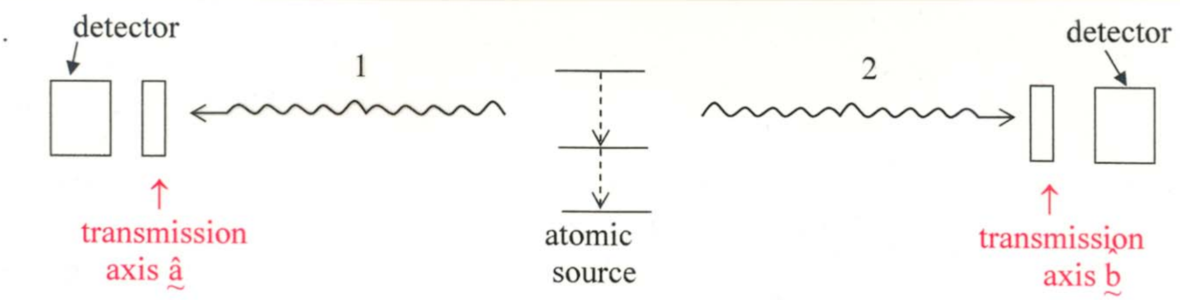
Then:

Prob. of transmission of 1 = ½.

But if 1 transmitted, then polarization of 2 is $\underset{\sim}{\hat{a}}$, so probability of transmission of polarizer set in direction $\underset{\sim}{\hat{b}} = \cos^2 \theta_{ab}$

(Malus's law)

$$P(\text{both transmitted}) = \tfrac{1}{2}\cos^2 \theta_{ab}$$



$$\text{Prob. (both detected)} = \frac{1}{2}\cos^2\theta_{ab} = \frac{1}{4}(1+\cos 2\theta_{ab})$$

"Isotropic mixture" $\Rightarrow \frac{1}{4}(1+\frac{1}{2}\cos 2\theta_{ab})$

$$\text{Prob. (neither detected)} = \frac{1}{2}\cos^2\theta_{ab}$$

$$\left.\begin{array}{l}\text{Prob. (1 detected, 2 not)}\\ \text{Prob. (2 detected, 1 not)}\end{array}\right\} = \frac{1}{2}\sin^2\theta_{ab}$$

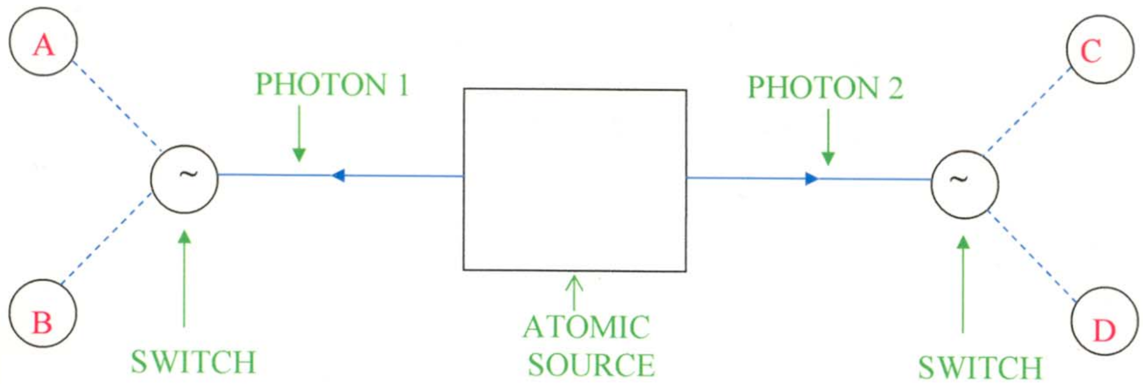
THESE ARE THE PREDICTIONS OF STANDARD QUANTUM MECHANICS. CAN THEY BE EXPLAINED BY A CLASSICAL PROBABILISTIC THEORY?

Df: If for a given pair, with polarizer 1 set at $\hat{a}$, photon 1 is detected, df. $A \equiv +1$: if rejected, $A \equiv -1$. Similarly with polarizer 2 set at $\hat{b}$, if photon 2 detected, df. $B \equiv +1$: if rejected, then $B \equiv -1$. Then above is equivalent to the statement that for the average over the ensemble of pairs,

$$\langle AB\rangle = \cos^2\theta_{ab} - \sin^2\theta_{ab} = \cos 2\theta_{ab} \qquad \text{["Mixture": } \tfrac{1}{2}\cos 2\theta_{ab}\text{]}$$

as special cases, for $\theta_{ab} = 0$ $\langle AB\rangle = +1$, and for $\theta_{ab} = \pi/2$. $\langle AB\rangle = -1$. (EPR). These two special cases can be accounted for by a classical probabilistic model. But…

EXPERIMENTS ON CORRELATED PHOTONS



((A) ≡ , etc.)

transm. axis = $\underset{\sim}{a}$

DEFINITION: If photon 1 is switched into counter “A

If counter “A” clicks, A = + 1 (DF.)

If counter “A” does not click, A = – 1 (DF.)

NOTE:

If photon 1 switched into counter “B”, then A is NOT DEFINED.

Experiment can measure

$\langle AC\rangle_{exp}$ on one set of pairs (1→ “A”, 2 → “C”)
$\langle AD\rangle_{exp}$ on another set of pairs (1 → “A”, 2 → “D”)
etc.

Of special interest is

$$K_{exp} \equiv \langle AC\rangle_{exp} + \langle AD\rangle_{exp} + \langle BC\rangle_{exp} - \langle BD\rangle_{exp}$$

for which Q.M. makes clear predictions.

POSTULATES OF "OBJECTIVE LOCAL" THEORY:

(1) Local causality

(2) Induction

(3) Microscopic realism OR macroscopic "counter-factual definiteness"

BELL'S THEOREM

1. (3) → For each photon 1, EITHER A = + 1 OR A = – 1, independently of whether or not A is actually measured.

2. (1) → Value of A for any particular photon 1 unaffected by whether C or D measured on corresponding photon 2. : etc.

3. ∴ For each pair, quantities AC, AD, BC, BD exist, with A, B, C, D, = ± 1 and A the same in (AC, AD) (etc.)

4. Simple algebra then → for each pair, AC + AD+ BC– BD ≤ 2

5. Hence for a single ensemble,

$$\langle AC\rangle_{ens} + \langle AD\rangle_{ens} + \langle BC\rangle_{ens} - \langle BD\rangle_{ens} \leq 2$$

6. (2) → $\langle AC\rangle_{exp} = \langle AC\rangle_{ens}$, hence the measurable quantity

$$K_{exp} \equiv \langle AC\rangle_{exp} + \langle BC\rangle_{exp} + \langle BC\rangle_{exp} - \langle BD\rangle_{exp}$$

satisfies

$$\boxed{K_{exp} \leq 2, \quad \text{Obj. Local Theory}}$$

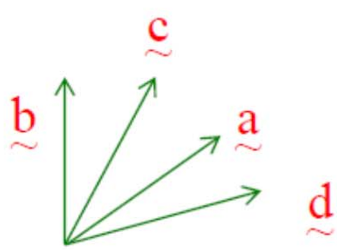
OBJECTIVE LOCAL THEORY: $\underline{K_{exp} \le 2}$.

QM: If polarizer settings are $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$, $\underset{\sim}{d}$

then e.g. for a 0^+ transition predict

$$\langle AC \rangle = \cos(2\theta_{\underline{a}\cdot\underline{c}}), \text{ etc.}$$

$\Rightarrow$ for



QM predicts (ideal case)

$$\underline{K_{exp} = 2\sqrt{2}}$$

$\Rightarrow$ Exptl. Predictions of QM incompatabile with those of an

theory embodying

- Local causality
- Induction
- Macroscopic counter-factual definiteness

1. "It is a fact that <u>either</u> A would have clicked <u>or</u> A wo
   have clicked"
2. "<u>Either</u> it is a fact that A would have clicked, <u>or</u> it is
   that A would not have clicked"

IDEA OF "ENTANGLEMENT":

System composed of 2 (separated) subsystems 1 and 2:

$$\Psi = \Psi(1,2) \quad \text{[general]}$$

(a) $\Psi(1,2) = \chi(1)\phi(2)$ product, nonentangled

"Properties" of 1 descr. by $\chi(1)$

"Properties" of 2 descr. by $\phi(2)$

Complete information on system obtainable by making measurements on subsystems separately.

(Note: classical "mixture" of non-entangled states are themselves non-entangled, e.g. mixture of ($\uparrow_1\downarrow_2$ and $\downarrow_1\uparrow_2$ is non-entangled).

(b) $\Psi(1,2) \neq \chi(1)\,\phi(2)$

e.g. (2 particles of spin ½): entangled

$$\Psi(1,2) = \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$$

($S = 0$) ("EPR pair")

Subsystems 1 and 2 do not "possess" individual properties (Bell's theorem).

Complete information on system obtainable only by correlated measurements on 1 and 2

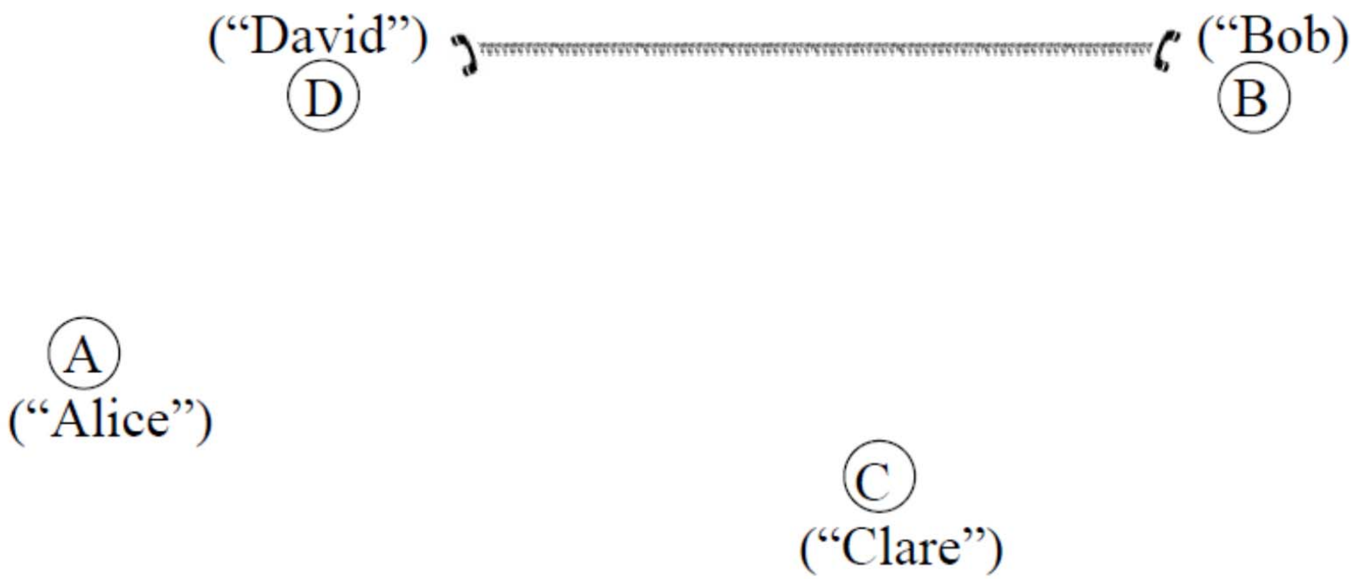
INFORMATION "STORAGE" IS NONLOCAL!

Note: Spin$\frac{1}{2}$ $\cong$ photon polarization:

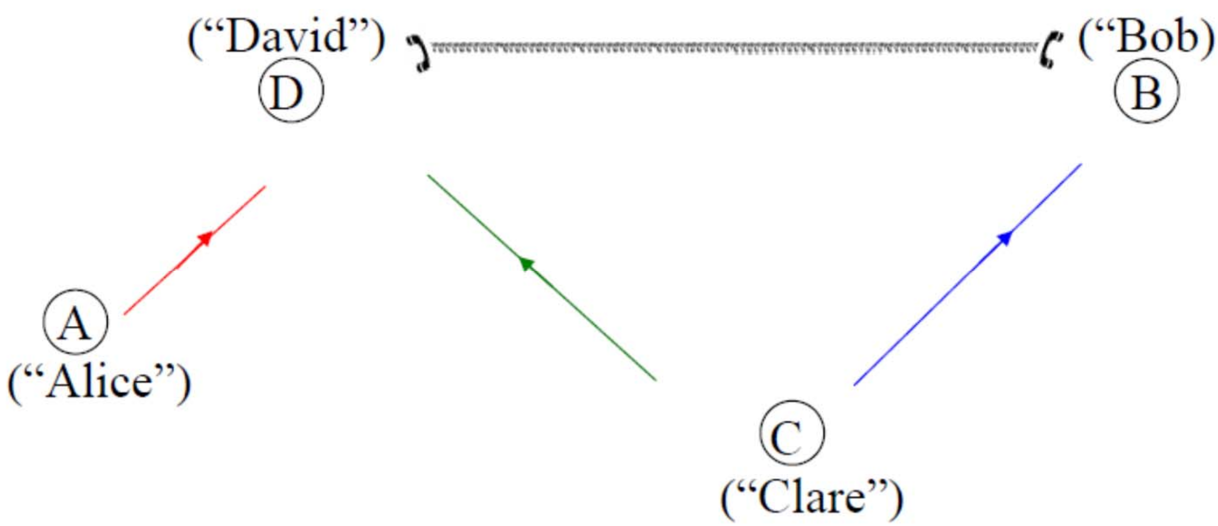
$$\uparrow \;\cong\; \uparrow \qquad \downarrow \;\cong\; \longrightarrow$$

1. Quantum "Teleportation"

(e.g. of state of particle of spin ½)

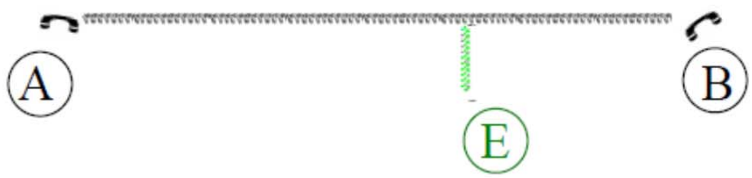


Rules of the game: Alice is to transmit to Bob an arbitrary (in general unknown to her) state $|\Psi>$ of a particle of spin ½, without direct physical contact (but A (or D) may communicate with B, e.g. by a <u>classical</u> phone line).



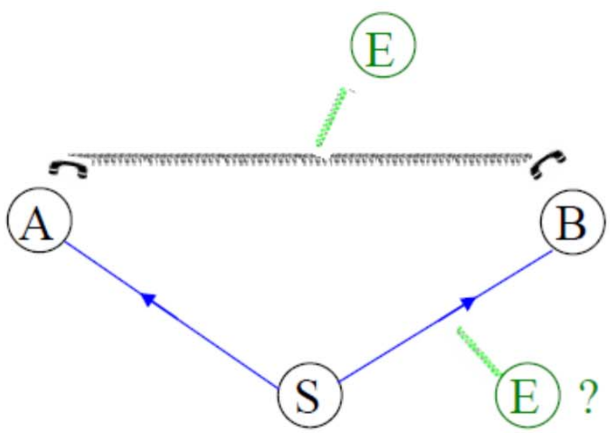
Solution: C emits "EPR pair." (S = 0). D then measures combined state of R and G spins (photons). If D finds S = 0, then spin conservation ⇒ state received by B = that sent by A, (and D phones B to tell him so). If D finds S ≠ 0, can "rotate" into S = 0. Then B must perform inverse of this rotation (and D so instructs him).

2. Quantum Cryptography:



Key distribution problem: Alice must be able (a) to communicate the "key" (a string of 1's and 0's) to Bob, and moreover know if Eve is listening in. Classically, no 100% secure way of ensuring this known.

Solution:



S emits a string of EPR pairs (S = 0). A measures in basis , or in basis , in a random way: B measures similarly, also at random. At the end, A and B inform one another by phone which basis they have used for each measurement, discard those for which they used different bases, and compare notes on a subset of the rest. If they always agree, they can be sure of no eavesdropping, and so use the rest for the key.

If Eve tries to "listen in" on the <u>quantum</u> channel…

3. Quantum Computation:

EX: particles of spin ½.

A single particle of spin ½ in a pure state is parametrized by 2 independent variables, e.g. corresponding to the angles (θ, φ) made by its spin with the z- and x-axes.

A collection of N particles of spin ½ in a product state is parametrized similarly by 2 independent variables for each, i.e. 2N in total.

But the same collection of N particles in a generic entangled state requires for its parametrization 2^N variables!

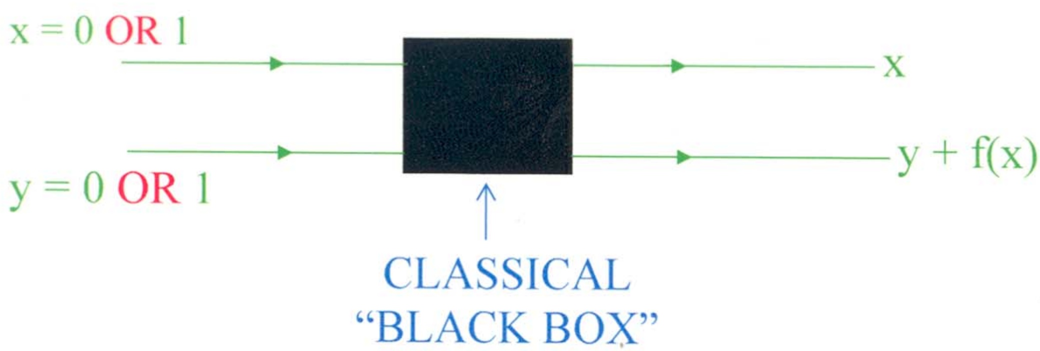
e.g. (N = 3)

$$\Psi(1,2,3) = a_1|\uparrow_1\uparrow_2\uparrow_3> + a_2|\uparrow_1\uparrow_2\downarrow_3> + a_3|\uparrow_1\downarrow_2\uparrow_3>$$
$$\ldots + a_8\,|\downarrow_1\downarrow_2\downarrow_3>$$

⇒ MASSIVELY PARALLEL COMPUTATION!

(Shor, 1994: factorization of N-digit no. takes time which for classical computer is exponential in N, but for quantum computer is power-law).

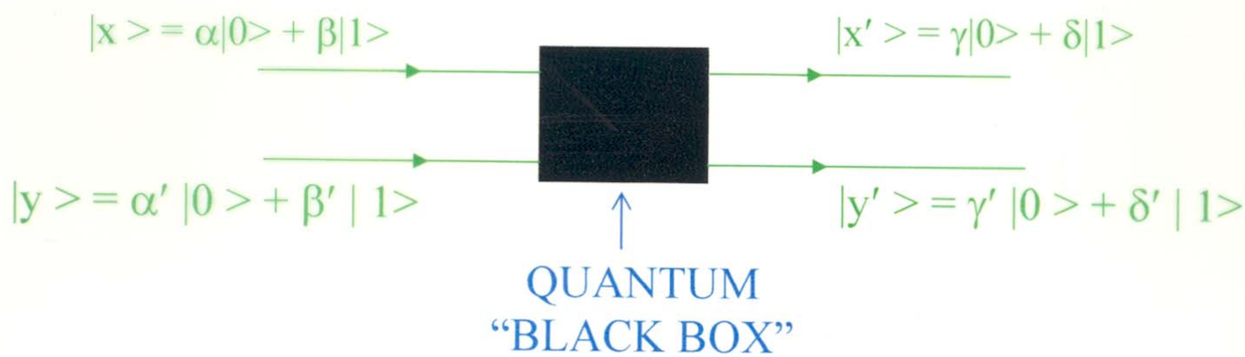
DEUTSCH'S ALGORITHM (SIMPLIFIED VERSION)*



Since x = {0,1}, only 4 possible mappings $f: x \rightarrow x$

Question: Does $f(0) = f(1)$?

Classically, need to input x = 0 (y can be 0 or 1) and measure output of lower bit, then same with x = 1. (2 measurements)

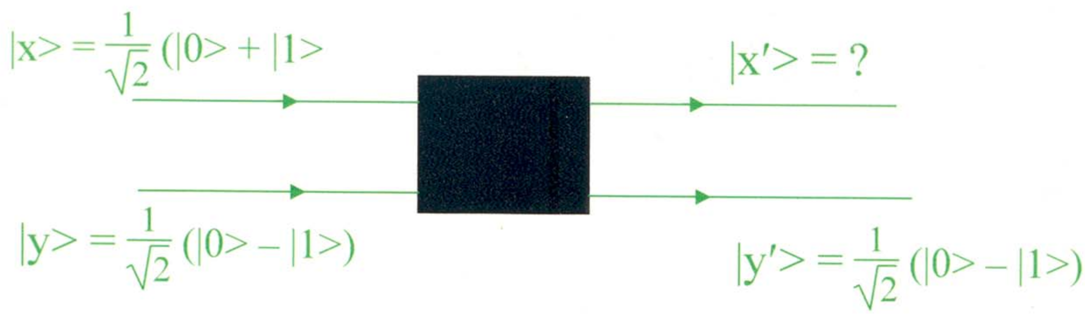


Input: $|x\rangle = 2^{-1/2}\,(|0\rangle + |1\rangle)$

$|y\rangle = 2^{-1/2}\,(|0\rangle - |1\rangle$

$|0,1\rangle \rightarrow |0,1 + f(0)\rangle$ etc.

*after Nielsen + Chuang, Q. Comp. + Q. Inf., § 1.4.3



If $f(0) = f(1)$, then $|x'> = \frac{1}{\sqrt{2}}(|0> + |1>) \equiv | + >$

If $f(0) \neq f(1)$, then $|x'> = \frac{1}{\sqrt{2}}(|0> - |1>) \equiv | - >$

By appropriate unitary transformation, convert

$$| + > \rightarrow |0>$$

$$| - > \rightarrow |1>$$

Then measure x: (1 measurement!)

If $x = 0$, then $f(0) = f(1)$

If $x = 1$, then $f(0) \neq f(1)$

[NB: Deutsch's algorithm itself does <u>not</u> exploit entanglement, but almost all more sophisticated algorithms do]

(SIMPLEST) DESIGN FOR A QUANTUM COMPUTER:

couple together N 2-state systems
("QUBITS") in such a way that we
can reliably perform "1-qubit" operations
(non-entangling) and "2-qubit" operations (entangling)

e.g. Heisenberg interaction $J\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2$ induces

$$\underset{\text{non-entangled}}{\uparrow_1\downarrow_2} \Longrightarrow \underset{\text{entangled}}{2^{-1/2}(\uparrow_1\downarrow_2 + i\downarrow_1\uparrow_2)}$$

Principal requirements for qubit:

- 2 states only
- easy initialization and readout
- scalable
- * decoherence-free

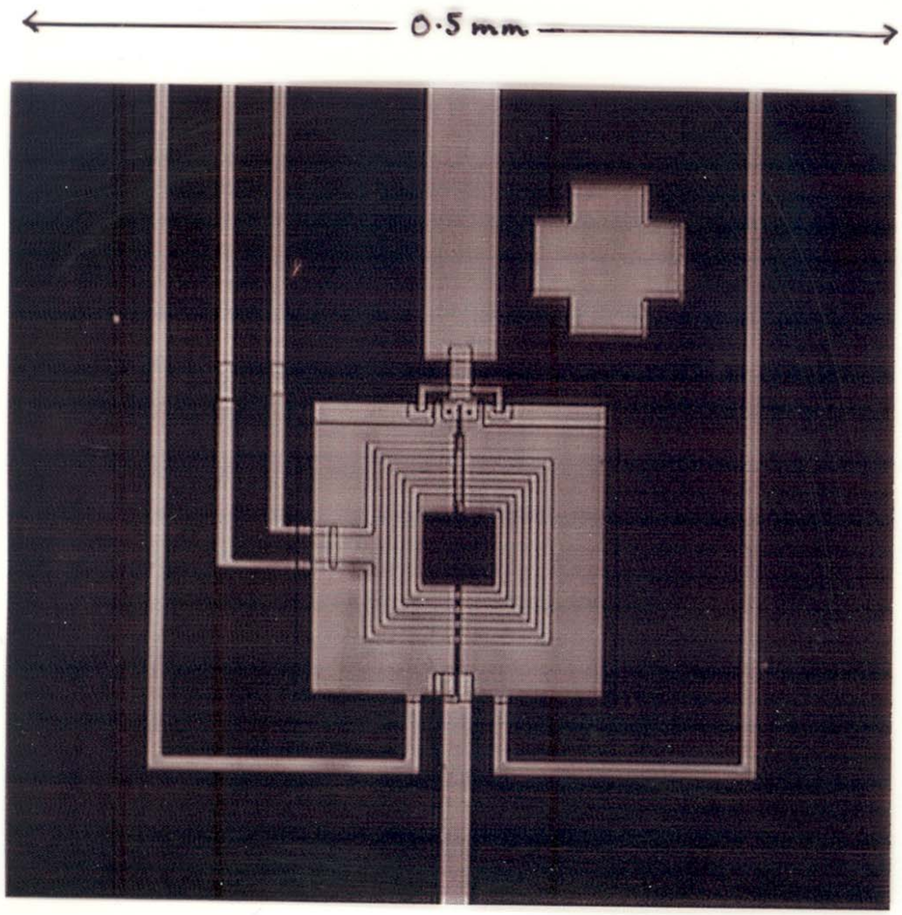
"Figure of merit" for (lack of) decoherence:

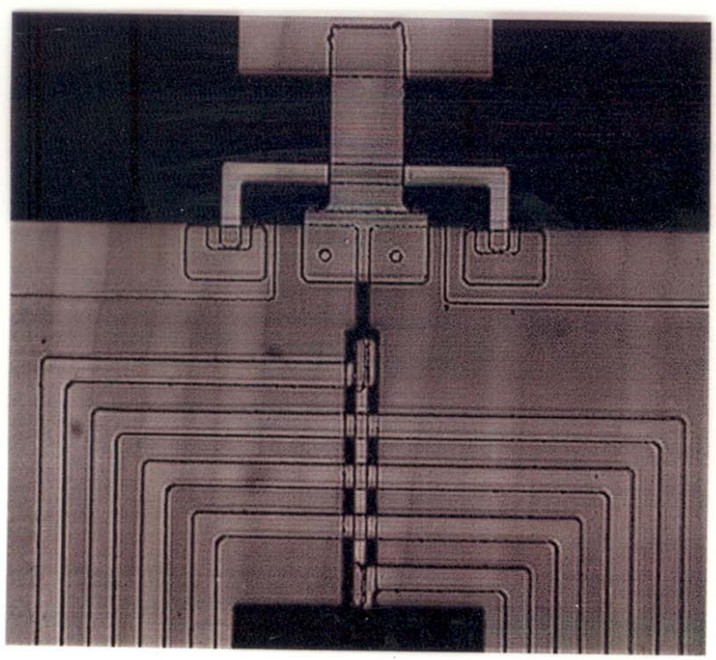
$$Q_\varphi \equiv \omega_o \, T_\varphi$$

ω_o: char. freq. of 2-state system

T_φ: "dephasing" time

Current "designs" require $Q_\varphi \gtrsim 10^4$


0.5 mm



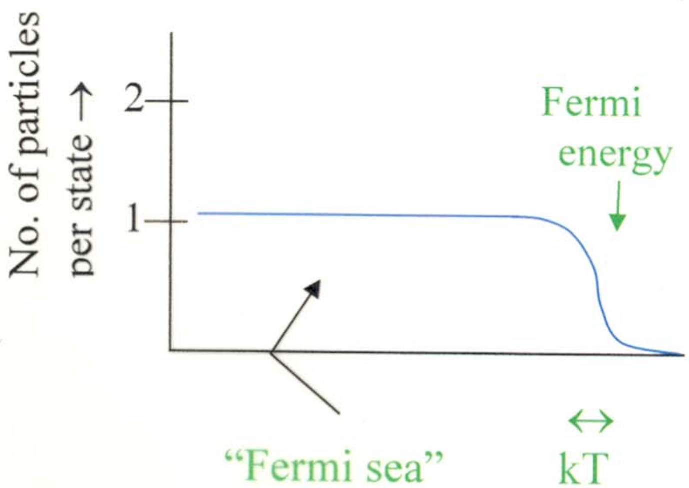
PHYSICS OF SUPERCONDUCTIVITY

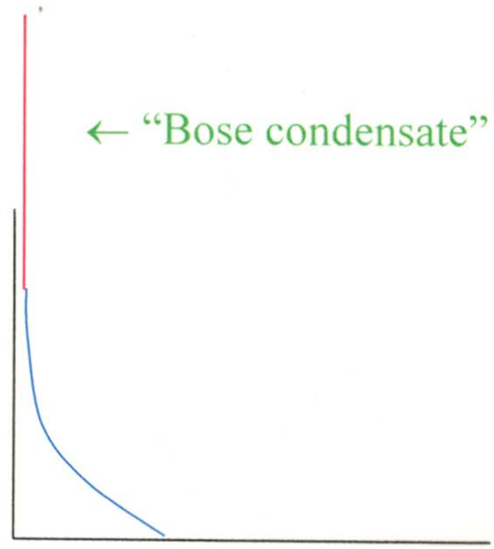
"Spin" of elementary particles $= \frac{n}{2}\,\hbar$

$0, 1, 2....$ bosons

$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}....$ fermions

At low temperatures:





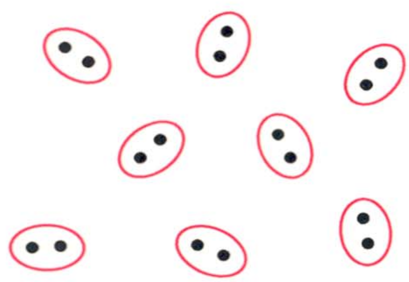
Electrons in metals: spin ½ ⇒ fermions

But a compound object consisting of an even no. of fermions has spin 0, 1, 2 … ⇒ boson.

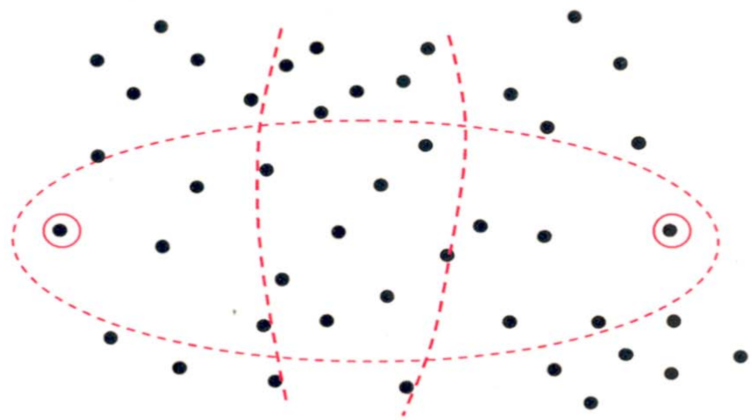
(Ex: 2p + 2n + 2e = ^{4}He atom)

⇒ can undergo Bose condensation

Pairing of electrons:



"di-electronic molecules"

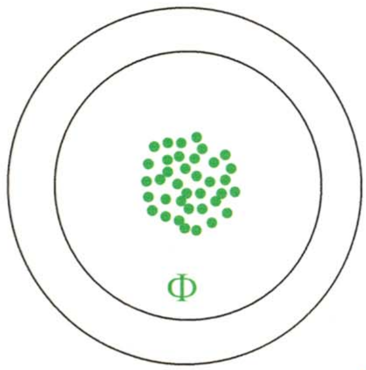


Cooper Pairs

In simplest ("BCS") theory, Cooper pairs, once formed, must automatically undergo Bose condensation!

$\Rightarrow$ must all do exactly the same thing at the same time (also in nonequilibrium situation)

SUPERCONDUCTING RING IN EXTERNAL MAGNETIC FLUX:



Quantization condition for "particle" of charge 2e (Cooper pair):

integer

$$K \equiv \oint \mathbf{v} \cdot \mathbf{dl} = \frac{h}{2m}(n - \Phi/\Phi_o)$$

$E \propto K^2$

"circulation"

"flux quantum" h/2e

$E \propto K^2$

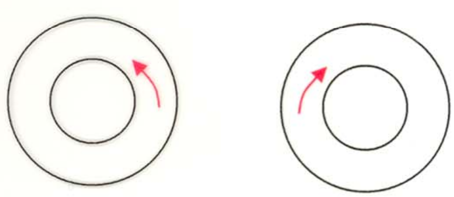
$\propto (n - \Phi/\Phi_o)^2)$

A. $\Phi = 0$: groundstate unique ($n = 0$)

$\Rightarrow$ all pairs at rest.

B. $\Phi = 1/2\ \Phi_o$: groundstate doubly degenerate ($E \propto (n - 1/2)^2$)

($n = 0$ or $n = 1$)



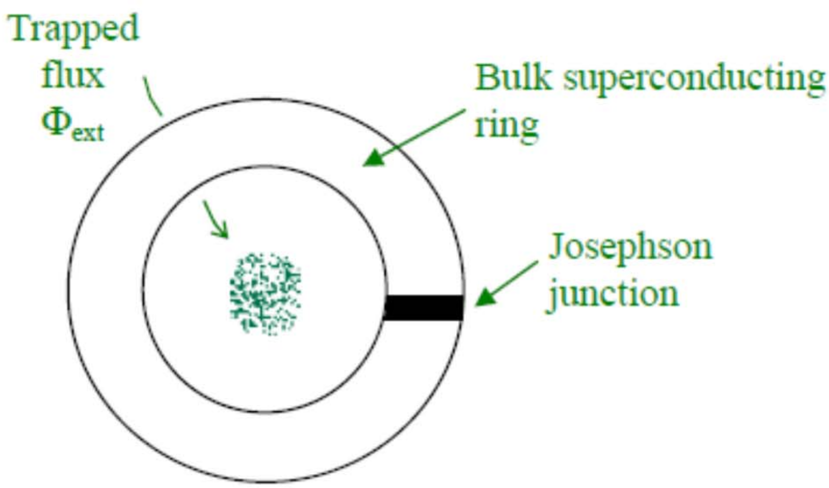
Either all pairs rotate clockwise

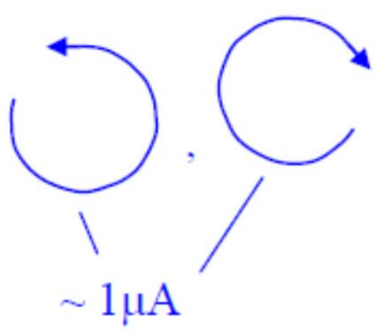
Or all pairs rotate anticlockwise

Note: state with 50% and 50%

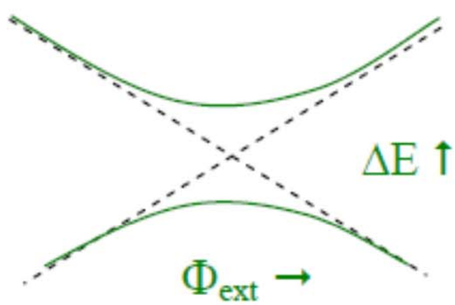
strongly forbidden by energy considerations

Josephson "qubit"





$\Psi = 2^{-1/2}\ (|↺> + |↻>)$



Evidence: (a) spectroscopic:
(SUNY, Delft 2000)

(b) real-time oscillations (like NH_3)

between ↺ and ↻

(Saclay 2002, Delft 2003) (Q_φ ~ 50-100)