Some Thoughts on the Prospects for Topological Quantum Computing

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based on work supported by the National Science Foundation
under grant no. NSF-EIA-01-21568

TOPOLOGICAL QUANTUM COMPUTING/MEMORY

Qubit basis. $|\uparrow\rangle, \ |\downarrow\rangle$

$$|\Psi\rangle = \alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle$$

To preserve, need (for "resting" qubit)

$$\hat{H} \propto \hat{1} \qquad \text{in } |\uparrow\rangle,\ |\downarrow\rangle \text{ basis}$$

$$(\hat{H}_{12} = 0 \Rightarrow \ \text{"}T_1 \to \infty\text{"}: \ \hat{H}_{11} - \hat{H}_{22} = \text{const} \Rightarrow \text{"}T_2 \to \infty\text{"})$$

on the other hand, to perform (single-qubit) operations, need to impose nontrivial $\hat{H}$.

$\Rightarrow$we must be able to do something Nature can't.

(ex: trapped ions: we have laser, Nature doesn't!)

<u>Topological protection:</u>

would like to find d–(>1) dimensional Hilbert space within which (in absence of intervention)

$$\hat{H} = (const.)\bullet\hat{1} + o\,(e^{-L/\xi})$$

size of system

microscopic length

How to find degeneracy?

Suppose $\exists$ two operators $\hat{\Omega}_1, \hat{\Omega}_2$ s.t.

$$[\hat{H},\hat{\Omega}_1] = [\hat{H},\hat{\Omega}_2] = 0 \qquad (\text{and } \hat{\Omega}_1,\hat{\Omega}_2 \text{ commute with b.c's})$$

but

$$[\hat{\Omega}_1,\hat{\Omega}_2] \neq 0 \qquad (\text{and } \hat{\Omega}_1\,|\,\psi> \neq 0)$$

Example of Topologically Protected State: FQH System on Torus (Wen and Niu, PR B **41**, 9377 (1990))

Reminders regarding QHE:

2D system of electrons, $B \perp$ plane

Area per flux quantum = $(h/eB) \Rightarrow$ df.

$$\ell_M \equiv (\hbar / eB)^{1/2} \leftarrow \text{"magnetic length"}$$

$(\ell_M \sim 100\dot{A}$ for B = 10 T)

"Filling fraction" ≡ no. of electrons/flux quantum ≡ ν

"FQH" when $\nu = \underbrace{p/q}_{\text{incommensurate integers}}$

Argument for degeneracy: (does not need knowledge of w.f.)

can define operators of "magnetic translations"

$\hat{T}_x(\boldsymbol{a}), \hat{T}_y(\boldsymbol{b})$ (≡ translations of all electrons through **a**(**b**) × appropriate phase factors). In general $[\hat{T}_x(\boldsymbol{a}), \hat{T}_y(\boldsymbol{b})] \neq 0$

In particular, if we choose ← no. of flux quanta

$$\boldsymbol{a} = \boldsymbol{L}_1 / N_s, \;\; \boldsymbol{b} = \boldsymbol{L}_2 / N_s \qquad (= L_1 L_2 / 2\pi\ell^2)$$

then $\hat{T}_1, \hat{T}_2$ commute with b.c.'s (?) and moreover

$$\hat{T}_1 \hat{T}_2 = \hat{T}_2 \hat{T}_1 \exp - 2\pi i \nu$$

But the o. of m. of $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\ell_M \cdot (\ell_M / L) \equiv \ell_{osc} \ll \ell_M$, and $\Rightarrow 0$ for $L \to \infty$. Hence to a very good approximation,

$$[\hat{T}_1, \hat{H}] = [\hat{T}_2, \hat{H}] = 0 \qquad (*)$$

so since $[\hat{T}_1, \hat{T}_2] \neq 0$

must ∃ more than 1 GS (actually q).

Corrections to (*): suppose typical range of (e.g.) external potential V(**r**) is ℓ_o, then since $|\psi\rangle$'s oscillate on scale ℓ_{osc},

$$\langle \psi_1 | \hat{H} | \psi_2 \rangle \sim \exp - \ell_o / \ell_{osc} \sim \exp - L/\xi$$

$$(+ \text{ const. } \hat{1}) \qquad \equiv \ell_M^2 / \ell_o$$

TOPOLOGICAL PROTECTION AND ANYONS

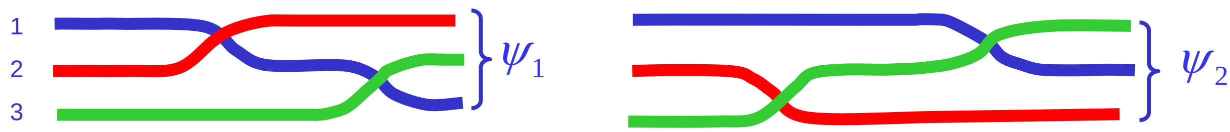
Anyons (df): exist only in 2D

$$\Psi(1,2) = \exp(2\pi i\alpha)\Psi(2,1) \equiv \hat{T}_{12}\Psi(1,2)$$

(bosons: α = 1, fermions: α = ½)

abelian if $\hat{T}_{12}\hat{T}_{23} = \hat{T}_{23}\hat{T}_{12}$ (ex: FQHE)

nonabelian if $\hat{T}_{12}\hat{T}_{23} \neq \hat{T}_{23}\hat{T}_{12}$, i.e., if



$$\psi_1 \neq \psi_2$$

("braiding statistics")

Nonabelian statistics* is a sufficient condition for topological protection:

[not necessary, cf. FQHE on torus]

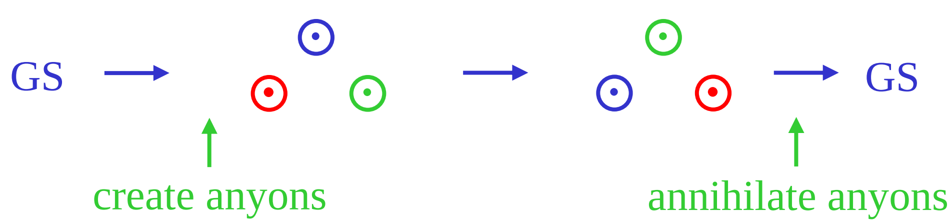
(a) state containing n anyons, $n \geq 3$:

$$[\hat{T}_{12}, \hat{H}] = [\hat{T}_{23}, \hat{H}] = 0$$

$$[\hat{T}_{12}, \hat{T}_{23}] \neq 0$$

$\Rightarrow$ space must be more than 1D.

(b) groundstate:



annihilation process inverse of creation $\Rightarrow$

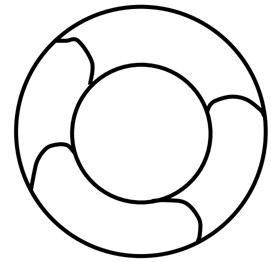
GS also degenerate.

*plus gap for anyon creation

Nonabelian statistics may (depending on type) be adequate for (partially or wholly) topologically protected quantum computation

Specific Models with Topological Protection

1. FQHE on torus



Obvious problems:

(a) QHE needs GaAs–AlGaAs or Si MOSFET: how to "bend" into toroidal geometry?

QHE observed in (planar) graphene (but not obviously "fractional"!): bend C nanotubes?

(b) Magnetic field should everywhere have large compt $\perp$ to surface: but div **B** = 0 (Maxwell)!

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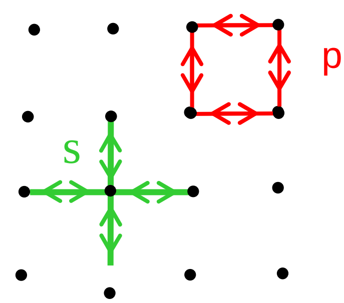
2. Spin Models (Kitaev et al.) (adv: exactly soluble)

(a) "Toric code" model

Particles of spin ½ on lattice



$$\hat{H} = -\sum_s \hat{A}_s - \sum_p \hat{B}_p$$

$$\hat{A}_s \equiv \prod_{j\varepsilon s} \hat{\sigma}_j^x \,, \quad \hat{B}_p \equiv \prod_{j\varepsilon p} \hat{\sigma}_j^z$$

(so $[\hat{A}_s, \hat{B}_p] \neq 0$ in general)

Problems:

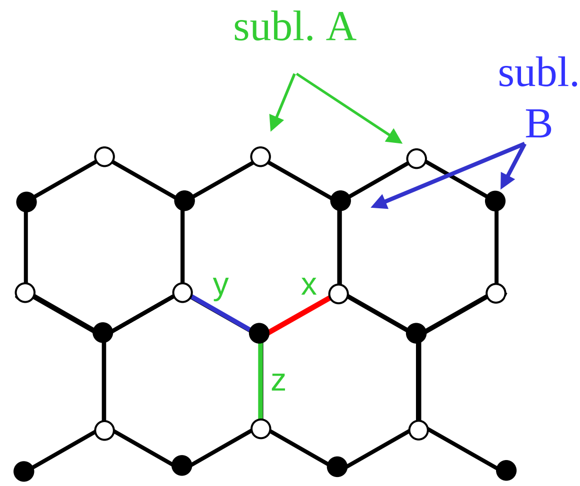
(a) toroidal geometry required (as in FQHE)

(b) apparently v. difficult to generate Ham^n physically

Spin Models (cont.)

(b) Kitaev "honeycomb" model

Particles of spin ½ on honeycomb lattice (2 inequivalent sublattices, A and B)



$$\hat{H} = -J_x \sum_{x-links} \hat{\sigma}_j^x \hat{\sigma}_k^x - J_y \sum_{y-links} \hat{\sigma}_j^y \hat{\sigma}_k^y - J_z \sum_{z-links} \hat{\sigma}_j^z \hat{\sigma}_k^z$$

nb: spin and space axes independent

Strongly frustrated model, but exactly soluble.*

Sustains nonabelian anyons with gap provided

$$|J_x| \le |J_y| + |J_z|, \; |J_y| \le |J_z| + |J_x|,$$
$$|J_z| \le |J_x| + |J_y| \qquad \text{and } \mathscr{H} \neq 0$$

(in opposite case anyons are abelian + gapped)

Advantages for implementation:

(a) plane geometry (with boundaries) is OK

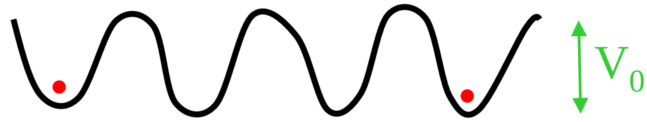
(b) $\hat{H}$ bilinear in nearest-neighbor spins

(c) permits partially protected quantum computation.

* A. Yu Kitaev, Ann. Phys. 321,2 (2006)
H-D. Chen and Z. Nussinov, cond-mat/070363 (2007)
(etc. …)

Can we Implement Kitaev Honeycomb Model?

One proposal (Duan et al., PRL *91*, 090492 (2003)): use optical lattice to trap ultracold atoms



Optical lattice:

3 counterpropagating pairs of laser beams create potential, e.g. of form

$$V(\boldsymbol{r}) = V_o(\cos^2 kx + \cos^2 ky + \cos^2 kz)$$

($2\pi/\lambda$ laser wavelength)

in 2D, 3 counterpropagating beams at 120° can create honeycomb lattice (suppress tunnelling along z by high barrier)

For atoms of given species (e.g. ^{87}Rb) in optical lattice 2 characteristic energies:

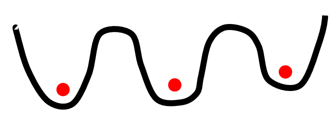
interwell tunnelling, t $(\sim e^{-\text{const.}\sqrt{V_0}})$

intrawell atomic interaction (usu. repulsion) U

For 1 atom per site on average:

if t » U, mobile ("superfluid") phase

if t « U, "Mott-insulator" phase
(1 atom localized on each site)



If 2 hyperfine species ($\cong$ "spin $-1/2$" particle), weak intersite tunnelling $\Rightarrow$ AF interaction

$$\hat{H}_{AF} = \sum_{nn} J \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j \qquad J = t^2/U$$

(irrespective of lattice symmetry).

So far, isotropic, so not Kitaev model. But …

If tunnelling is different for $\uparrow$ and $\downarrow$, then H'berg Hamiltonian is anisotropic: for fermions,

$$\hat{H}_{AF} = \frac{t_\uparrow^2 + t_\downarrow^2}{2U} \sum_{nn} \hat{\sigma}_i^z \hat{\sigma}_j^z + \frac{t_\uparrow t_\downarrow}{U} \sum_{nn} (\hat{\sigma}_i^x \hat{\sigma}_j^x + \hat{\sigma}_i^y \hat{\sigma}_j^y)$$

$\Rightarrow$ if $t_\uparrow \gg_\downarrow$, get Ising-type int^n

$$H_{AF} = \text{const.} \sum_{nn} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

We can control $t_\uparrow$ and $t_\downarrow$ with respect to an arbitrary "z" axis by appropriate polarization and tuning of (extra) laser pair. So, with 3 extra laser pairs polarized in mutually orthogonal directions (+ appropriately directed) can implement

$$\hat{H} = J_x \sum_{x-bonds} \hat{\sigma}_i^x \hat{\sigma}_j^x + J_y \sum_{y-bonds} \hat{\sigma}_i^y \hat{\sigma}_j^y + J_z \sum_{z-bonds} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

$\equiv$ Kitaev honeycomb model

Some potential problems with optical-lattice implementation:

(1) In real life, lattice sites are inequivalent because of background magnetic trap $\Rightarrow$ region of Mott insulator limited, surrounded by "superfluid" phase.

(2) V. long "spin" relaxation times in ultracold atomic gases $\Rightarrow$ true groundstate possibly never reached.

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Other possible implementations: e.g. Josephson circuits (You et al., arXiv: 0809.0051)

THE FRACTIONAL QUANTUM HALL EFFECT: THE CASES OF $\nu = 5/2$ AND $\nu = 12/5$

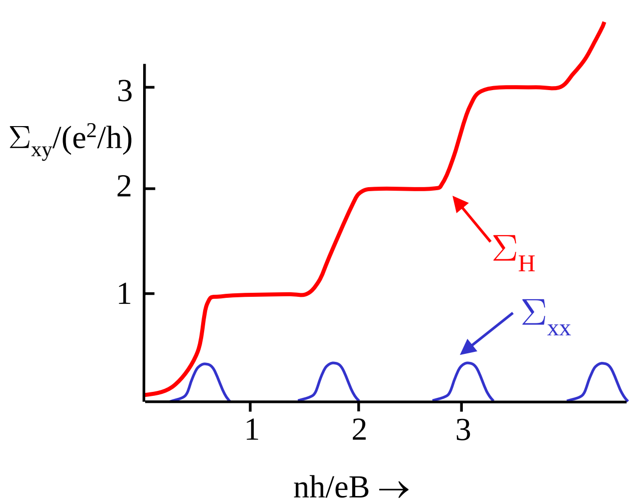
Reminder re QHE:

Occurs in (effectively) 2D electron system ("2DES") (e.g. inversion layer in GaAs – GaAlAs heterostructure) in strong perpendicular magnetic field, under conditions of high purity and low ($\lesssim$ 250 mK) temperature.

If df. $l_m \equiv (\hbar/eB)^{1/2}$ ("magnetic length") then area per flux quantum h/e is $2\pi l_m^2$, so no. of flux quanta $= A/2\pi l_m^2$ ($A \equiv$ area of sample). If total no. of electrons* is N_e, define

$$\nu \equiv N_e / N_\Phi \qquad \text{("filling factor")}$$

QHE occurs at and around (a) integral values of ν (integral QHE) and (b) fractional values p/q with fairly small ($\lesssim$ 13) values of q (fractional QHE). At ν'th step, Hall conductance Σ_{xy} quantized to $\nu e^2/\hbar$ and longitudinal conductance $\Sigma_{xx} \cong 0$



Nb: (1) Fig. shows IQHE only

(2) expts usually plot

$$R_{xy} \text{ vs } B\left(\propto \frac{1}{\nu}\right)$$

so general pattern is same but details different

* strictly, no./spin: valley (but see below)

SYSTEMATICS OF FQHE

FQHE is found to occur at and near $\nu = p/q$, where p and q are mutually prime intergers. By now, $\sim$ 50 different values of (p/q). Generally, FQHE with large values of q tend to be more unstable against disorder and temperature.

eg. plateaux narrower, $\rho_{xx} \not\to 0$

Possible approaches to identification of phases :

(a) analytic, trial wf (eg Laughlin)

(b) numerical, few-electron (typically $N \simeq 18$)

(c) via CFT $\leftarrow$ conformal field theory

(d) experiment :

alas, cannot usually measure much other than electrical props ! ideally, would at least like to know total spin of sample, but........

The simplest FQHE states (Laughlin states) : reminders

The Laughlin states have $p = 1, q =$ odd integer, i.e.

$$\nu = 1/(2m+1) \qquad , m = \text{integer (e.g. } \nu = \tfrac{1}{3}, \tfrac{1}{5}, \ldots\ldots)$$

These are well accounted for by the Laughlin w.f.

$$\Psi_N = \Pi^N_{i<j}(z_i - z_j)^q \exp - \textstyle\sum_i |z_i|^2/4l_m^2$$

$$q = \frac{1}{\nu} = 2m+1 \qquad\qquad (z \equiv x + iy)$$

Elementary excitations are quasiholes generated by multiplying GSWF by $\Pi^N_{i=1}(z_i - \eta_0)$ (hole at η_0). They have charge

$$e^* = \nu e$$

and are abelian anyons :

$$\Psi(1,2) = \exp i\pi\nu \Psi(2,1)$$

Fairly convincing evidence for fractional charge ($\nu = 1/3$), some evidence for fractional statistics.

THE ν = 5/2 STATE

First seen in 1987: to date the only even-denom. FQHE state reliably established* (some ev. for ν = 19/8). Quite robust: $\Sigma_{xy}/(e^2/h)$ = 5/2 to high accuracy, excluding e.g. odd-denominator values ν = 32/13 or 33/13*, and Σxx vanishes within exptl. accuracy. The gap Δ ~ 500 mK.

WHAT IS IT?

First question: is it totally spin-polarized (in relevant LL)? Early experiments showed that tilting ***B*** away from ⊥'r destroyed it ⇒ suggests spin singlet. But later experimental work, and numerics, suggests this may be ∵ tilted field changes orbital behavior and hence effective Coulomb interaction. So general belief is that it is totally spin-polarized (i.e. LLL ↑, ↓ both filled, *n* = 1 analog of ν = ½.

However the actual ν = ½ state does not correspond to a FQHE plateau. In fact the CF approach predicts that for this ν

composite fermion

$$N_\phi^{eff} = N_\phi - 2N_e = 0$$

and hence the CF's behave as a Fermi liquid: this seems to be consistent with experiment. If LLL ↑, ↓ both filled, this argt. should apply equally to ν = 5/2 (since $(N_e/N_\phi)_{eff} = 1/2$).

So what has gone wrong?

One obvious possibility†:

Cooper pairing of composite fermions!

since spins ||, must pair in odd-l state, e.g., p-state.

*except for ν = 7/2 which is the corr. state with *n* = 1, ↑ filled.
*Highest denominator seen to date ~19
† Moore & Read, Nuc. Phys. B **360**, 362 (1991): Greiter et al. **66**, 3205 (1991)

THE "PFAFFIAN" ANSATZ

Consider the Laughlin ansatz formally corresponding to ν = 1/2:

$$\psi_N^L = \Pi_{i<j}(z_i - z_j)^2 \exp -\Sigma_i |z_i|^2 / 4l_m^2 \quad (z_i = \underline{\text{electron}} \text{ coord.})$$

This cannot be correct as it is symmetric under $i \leftrightarrow j$. So must multiply it by an antisymmetric function. On the other hand, do not want to "spoil" the exponent 2 in numerator, as this controls the relation between the LL states and the filling.

Inspired guess (Moore & Read, Greiter et al.): (N = even)

$$\psi_N = \psi_N^{(L)} \times Pf\left(\frac{1}{z_i - z_j}\right)$$

$$Pf(f(ij)) \equiv f(12)f(34)\ldots - f(13)f(24)\ldots. + \ldots.(\equiv \text{ Pfaffian})$$

antisymmetric under *ij*

This state is the exact GS of a certain (not very realistic) 3-body Hamiltonian, and appears (from numerical work) to be not a bad approximation to the GS of some relatively realistic Hamiltonians.

With this GS, a single quasihole is postulated to be created, just as in the Laughlin state, by the operation

$$\psi_{qh} = \left(\Pi_{i=1}^N (z_i - \eta_0)\right) \bullet \psi_N$$

It is routinely stated in the literature that "the charge of a quasihole is –*e*/4", but this does not seem easy to demonstrate directly: the arguments are usually based on the BCS analogy (quasihold ↔ *h/2e* vortex, extra factor of 2 from usual Laughlin-like considerations) or from CFT.

conformal field theory

These excitations are nonabelian ("Ising") anyons.

IS THE $\nu = 5/2$ FQHE STATE REALLY THE PFAFFIAN STATE?

Problem: Several alternative identifications of the $\nu = 5/2$ state (331, partially polarized, "anti-pfaffian…."). Some are abelian, some not: all however predict $e^* = e/4$ [does this follow from general topological considerations?]. Numerical studies tend to favor the Pfaffian, but….

2 very recent experiments:

A. Dolev et. al., Nature **452** 829 (2008)

Shot-noise expt., similar to earlier ones on $\nu = 1/3$ FQHE state. Interpretation needs some nontrivial assumptions about the states neighboring the edge channels through which cond^n takes place.

Conclusion:

data consistent with $e^* = e/4$, inconsistent with $e^* = e/2$

unfortunately, doesn't discriminate between Pfaffian and other identifications.

Radu et. al., Science **320** 895 (2008)

Tunnelling expt., measures T-dependence of tunnelling current across QPC $\leftarrow$ quantum point contact. Fits to theory of Wen for general FQHE state, which involves 2 characteristic numbers, e^* and g: for Pfaffian, $e^* = e/4$, $g = ½$ (also for other nonabelian candidates: abelian candidates have $e^* = e/4$ *but* $g = 3/8$ *or* $1/8$).

Conclusion: best fit to date is

$$e^*/e = 0.17, \qquad g = 0.35$$

which is actually closer to the abelian (331) state ($g = 0.375$) than to the Pfaffian.

THE ν = 12/5 STATE

This state has so far seen in only one experiment*: it is quite fragile (short plateau, $R_{xx} \nrightarrow 0$). It could perfectly well be the *n* = 1 LL analog of the 2/5 state, which fits in the CF picture (*p* = 2, *m* = 1), and would of course be Abelian. Why should it be of special interest?

In 1999 Read & Rezayi speculated that the ν = 1/3 Laughlin state and Pfaffian ν = ½ state are actually the beginning of a series of "parafermion" states with

$$\nu = k/(k+2)$$

The ansatz for the wave function is

$$\Psi_{k:N} = \Sigma_{pES_M} \Pi_{0<r<s<N/k} \chi(z_{p(kr+1)} \ldots\ldots z_{p(k(r+1))} : z_{p(ks+1)} \ldots\ldots z_{(k(s+1))}$$

where

$$\chi(z_1 \ldots\ldots z_k : z_{k+1} \ldots\ldots z_{2k}) \equiv (z_1 - z_{k+1})(z_1 - z_{k+2})(z_2 - z_{k+2}) (z_2 - z_{k+3}) \ldots\ldots\ldots\ldots (z_k - z_{2k})(z_k - z_{k+1})$$

The state $\psi_{k:L}$ can be shown to be the exact groundstate of the (highly unrealistic!) Hamiltonian

$$H = \Sigma_{i<j<l<\ldots} \delta(z_i - z_j)\delta(z_j - z_l)\delta(z_l - z_m) \ldots\ldots (k+1) \text{ terms}$$

The quasiholes generated by this state have charge *e* = e/(k+2)* and are nonabelian for $k \geq 2$; for *k* = 3 they are Fibonacci anyons, which permit universal TQC.

Of course, the no. 12/5 ≠ *k/(k* + 2). However, it is possible that the ν = 12/5 state is the *n* = 1, particle-hole conjugate of ν = 3/5. In this context it is intriguing that the ν = 13/5 state has never been seen ….

How would we tell? Interference methods?

*Xia et al., PRL **93** 176809 (2003)

p-WAVE FERMI SUPERFLUIDS (in 2D)

Generically, particle-conserving wave function of a Fermi superfluid (Cooper-paired system) is of form

$$\Psi_N = \mathcal{N} \cdot \Big(\sum_{k,\alpha\beta} c_k a^+_{k\alpha} a^+_{-k\beta}\Big)^{N/2} |vac\rangle$$

e.g. in BCS superconductor

$$\Psi_N = \mathcal{N} \Big(\sum_k c_k a^+_{k\uparrow} a^+_{-k\downarrow}\Big)^{N/2} |vac\rangle -$$

Consider the case of pairing in a spin triplet, p-wave state (e.g. 3He-A). If we neglect coherence between ↑ and ↓ spins, can write

$$\Psi_N = \Psi_{N/2,\uparrow} \Psi_{N/2,\downarrow}$$

Concentrate on $\Psi_{N/2,\uparrow}$ and redef. N→ 2N.

$$\Psi_{N\uparrow} = \mathcal{N} (\Sigma c_k a^+_k a^+_{-k})^{N/2} |vac\rangle$$

suppress spin index

What is c_k?

KE measured from μ

Standard choice:

$$c_k = \exp - i\phi_k \left(\frac{1-\varepsilon_k/E_k}{1+\varepsilon_k/E_k}\right)^{1/2}$$

real factor

"p+ip"

$$\left(\varepsilon_k^2 + |\Delta_k|^2\right)^{1/2}$$

How does c_k behave for k→0? For p-wave symmetry, $|\Delta_k|$ must ∝ k, so $|c_k| \sim \varepsilon_F / |\Delta_k| \sim k^{-1}$

Thus the (2D) Fournier transform of c_k is ∝ $r^{-1} \exp - i\varphi \equiv z^{-1}$,

and the MBWF has the form

$$\Psi_N(z_1 z_2 \ldots z_N) = Pf\left(\frac{1}{z_i - z_j}\right) \times \text{ uninteresting factors}$$

Conclusion: apart from the "single-particle" factor

$\exp - \frac{1}{4\ell^2}\sum_j |z_j|^2$, MR ansatz for ν = 5/2 QHE is identical to the "standard" real-space MBWF of a (*p* + *ip*) 2D Fermi superfluid.

Note one feature of the latter:

if $$\hat{\Omega} \equiv \sum_k c_k a_k^+ a_{-k}^+, \quad c_k = |c_k| \exp - i\varphi_k$$

then $$[\hat{L}_{2,}\hat{\Omega}] = -\hbar\hat{\Omega}$$

z-component of ang. momentum

so $$\Psi_N \equiv \text{const. } \hat{\Omega}^N |vac\rangle$$

possesses ang. momentum –N$\hbar$/2, no matter how weak the pairing!

Now: where are the nonabelian anyons in the *p* + *ip* Fermi superfluid?

Read and Green (Phys. Rev. B *61*, 10217(2000)):
nonabelian anyons are zero-energy fermions bound to cores of vortices.

Consider for the moment a single-component 2D Fermi superfluid, with *p* + *ip* pairing. Just like a BCS (s-wave) superconductor, it can sustain vortices: near a vortex the pair wf, or equivalently the gap Δ(**R**), is given by

COM of Cooper pairs

$$\Delta(\boldsymbol{R}) \equiv \Delta(z) = \text{ const. } z$$

Since $|\Delta(\mathbf{R})|^2 \to 0$ for $\mathbf{R} \to 0$, and (crudely) $E_k(\boldsymbol{R}) \sim (\varepsilon_k^2 + |\Delta(\boldsymbol{R})|^2)^{1/2}$, bound states can exist in core. In the s-wave case their energy is $\sim\eta\, |\Delta_0|^2\, \varepsilon_F$, $\eta \neq 0$, so no zero-energy bound states.

What about the case of (*p* + *ip*) pairing? If we approximate

$$\Delta(\boldsymbol{R}, \boldsymbol{\rho}) = \Delta(R)\partial_\rho \delta(\boldsymbol{\rho})$$

relative coord.

∃ mode with u(**r**) = v*(**r**), E = 0

Now, recall that in general

$$\psi_{exc}(\mathbf{r}) = \left(u(r)\hat{\psi}^{\dagger}(r) + u(r)\hat{\psi}(r)\right)|0\rangle \equiv \hat{Q}(r)|0\rangle$$

But, if $u^*(r) = u(r)$, then $\hat{Q}^{\dagger}(r) \equiv \hat{Q}(r)$! i.e.

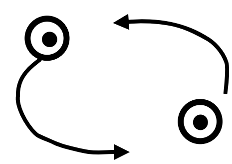
zero-energy modes are their own antiparticles

("Majorana modes")

⚠: This is true only for spinless particle/pairing of || spins (for pairing of anti || spins, particle and hole distinguished by spin).

Consider two vortices i, j with attached Majorana modes with creation ops. $\gamma_i \equiv \gamma_i^{\dagger}$.

What happens if two vortices are interchanged?*



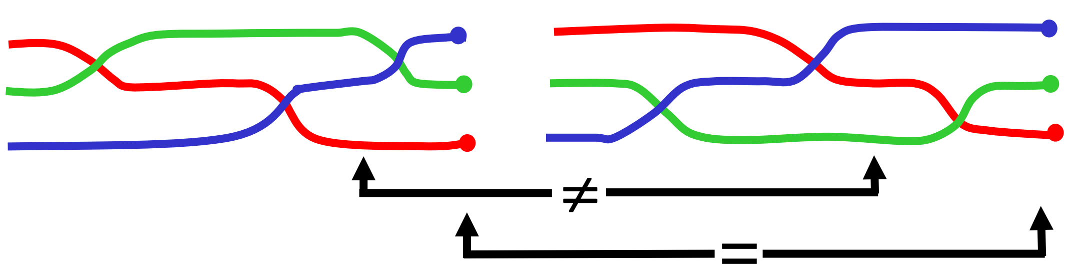
Claim: when phase of C. pairs changes by 2π, phase of Majorana mode changes by π (true for assumed form of u, v for single vortex). So

$$\gamma_i \to \gamma_j$$

$$\gamma_j \to -\gamma_i$$

more generally, if ∃ many vortices + w df $\hat{T}_i$ as exchanging i, $i+1$, then for $|i–j|>1$ $[\hat{T}_i, \hat{T}_j] = 0$, but

for $|i–j|=1$, $[\hat{T}_i, \hat{T}_j] \neq 0$, $\hat{T}_i\,\hat{T}_j\,\hat{T}_i = \hat{T}_j\,\hat{T}_i\,\hat{T}_j$ braid group!



* Ivanov, PRL *86*, 268 (2001)

How to implement all this?

(a) superfluid ^{3}He-A:
to a first approximation,

$$\Psi = \Psi_\uparrow \Psi_\downarrow, \quad \Psi_\uparrow = \left(\sum_k c_k a^+_{k\uparrow} a^+_{-k\uparrow} \right)^{N/2} |vac\rangle \text{ (etc.)}$$

$$c_k \sim |c_k| \exp i\varphi_k$$

so prima facie suitable.

Ordinary vortices ($\Delta_\uparrow(\boldsymbol{r}) \sim \Delta_\downarrow(\boldsymbol{r}) \sim z$) well known to occur. Will they do?

Literature mostly postulates half-quantum vortex ($\Delta_\uparrow(\boldsymbol{r}) \sim z$, $\Delta_\downarrow(\boldsymbol{r})$ = const., i.e. vortex in ↑ spins, none in ↓)

HQV's should be stable in ^{3}He-A under appropriate conditions (e.g. annular geom., rotation at $\omega \sim \omega_c/2$, $\omega_c \equiv \hbar/2mR^2$) sought but not found:

? ?

Additionally, would need a thin slab (how thin?) for it to count as "2D".

How would we manipulate vortices/quasiparticles (neutral) in ^{3}He-A?

What about charged case (p + ip superconductor)?

Ideally, would like 2D superconductor with pairing in (p + ip) state. Does such exist?

STRONTIUM RUTHENATE (Sr_2RuO_4)*

Strongly layered structure, anal. cuprates $\Rightarrow$ hopefully sufficiently "2D." Superconducting with $T_c \sim 1.5$ K, good type-II props. ($\Rightarrow$ "ordinary" vortices certainly exist).

$64 K question: is pairing spin triplet (p + ip)?
Much evidence* both for spin triplet and for odd parity ("p not s").

Evidence for broken T-reversal symmetry:

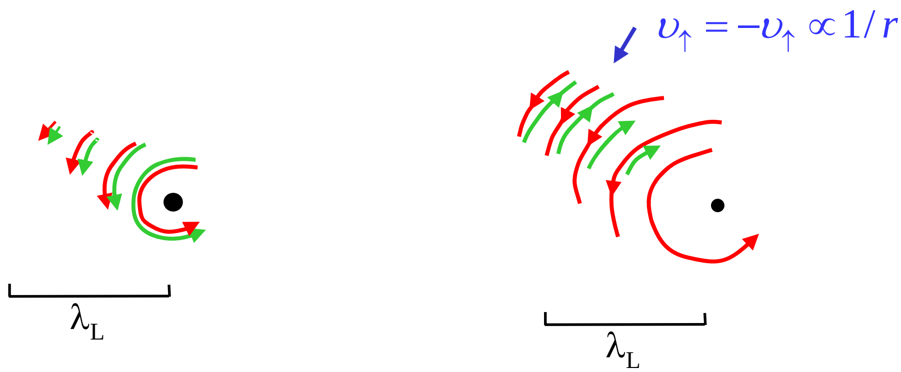
optical rotation (Xia et al. (Stanford), 2006)
Josephson noise (Kidwingira et al. (UIUC), 2006)

$\Rightarrow$"topology" of orbital pair w.f. probably ($p_x + ip_y$).

Can we generate HQV's in Sr_2RuO_4?

Problem:
in neutral system, both ordinary and HQ vortices have $1/r$ flow at ∞. $\Rightarrow$HQV's not specially disadvantaged. In charged system (metallic superconductor), ordinary vortices have flow completely screened out for $r \gtrsim \lambda_L$ by Meissner effect. For HQV's, this is not true:

London penetration depth



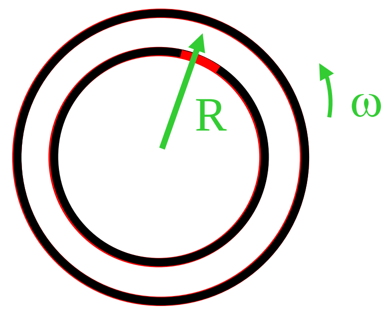
So HQV's intrinsically disadvantaged in Sr_2RuO_4.

*Mackenzie and Maeno, Rev. Mod. Phys. *75*, 688 (2003)

Problems:

(1) Is Sr_2RuO_4 really a (p + ip) superconductor?
If so, is single-particle bulk energy gap nonzero everywhere on F.S.?
Even if so, does large counterflow energy of HQV mean it is never stable?

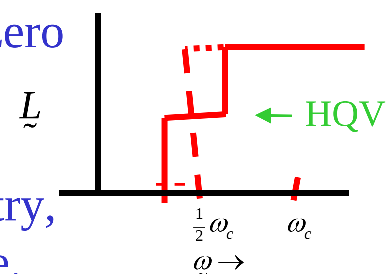
(2) Non-observation of HQV's in ^{3}He-A:
Consider thin annulus rotating at ang. velocity ω, and df. $\omega_c \equiv \hbar / 2mR^2$



At $\omega = \frac{1}{2}\omega_c$ exactly, the nonrotating state and the ordinary "vortex" (p-state) with both spins rotating are degenerate.

But a simple variational argument shows that barring pathology, there exists a nonzero range of ω close to $\frac{1}{2}\omega_c$ where the HQV is more stable than either!

In a simply connected flat-disk geometry, argument is not rigorous but still plausible.



Δ: Yamashita et al. (2008) do experiment in flat-disk geometry, find NO EVIDENCE for HQV!

Possible explanations:

(1) HQV is never stable (Kawakami et al., preprint, Oct 08)

(2) HQV did occur, but NMR detection technique insensitive to it.

(3) HQV is thermodynamically stable, but inaccessible in experiment.

(4) Nature does not like HQV's.

Problems (cont.)

More fundamental problem:

Does the existence of a "split E=0 DB fermion" survive the replacement of the scale-invariant gap fermion

$$\Delta(r,r') = \frac{\Delta_b}{k_F}\partial_r \delta(\underline{r} - \underline{r}')$$

by the true gap $\Delta(\underline{r} - \underline{r}')$?

Recall: real-space width of "MF" is

$$\ell \sim k_F^{-1}(R_o / \xi)$$

<u>but</u>, range of real-life $\Delta(\boldsymbol{r} - \boldsymbol{r}') \gtrsim k_F^{-1}$!

Possible clues from study of toy model

$$\hat{H} = \sum_{j=1}^{N-1} (-ta_j^+ a_{j+1} - i\Delta a_j^+ a_{j+1}^+ + H.c.) - \mu \sum_{j=1}^{N} a_j^+ a_j$$

as f'n of ratios Δ/t and μ/t, <u>taking proper account of boundary conditions</u>.

For Δ=t, μ=0   2 MF's exist at ends of chain

For Δ = 0, any t/μ, trivially soluble, no MF's or anything else exotic.

Where and how does crossover occur? (cf. Lu and Yip, Oct. 2008)

$64,000 question: in real life, are the MF's a science fiction?