Spontaneous Symmetry Breaking: Its Successes, Its Limitations, and Its Pitfalls

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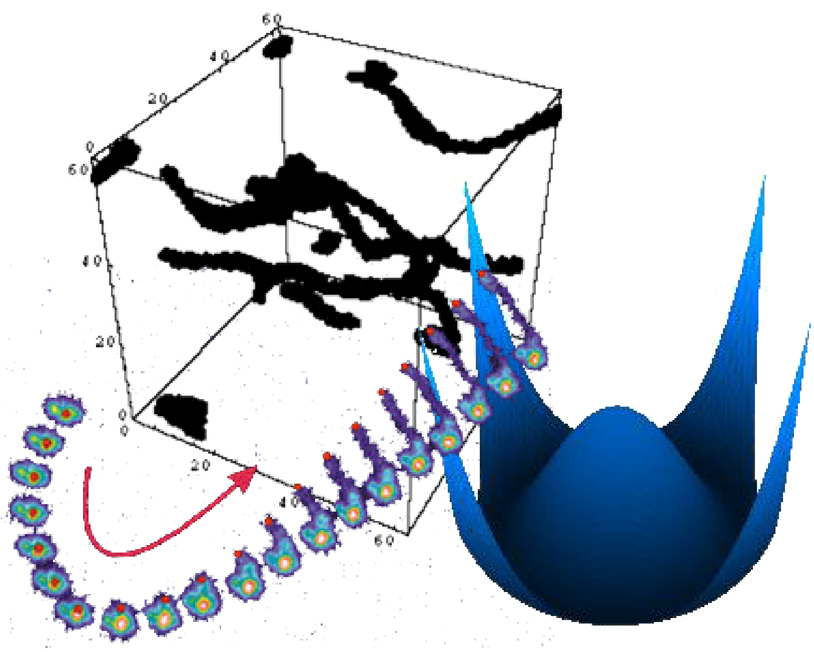


Broken Symmetry

Hamiltonian (Lagrangian) of system is invariant under various symmetry operations.

Examples:

(1) Nonrelativistic CM system, e.g. gas of atoms of spin ½: (zero magnetic field)

$$\hat{H} = -\frac{1}{2M}\sum_i \nabla_i^2 + \frac{1}{2}\sum_{i\neq j} V(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$$

invariant under:

$$\overbrace{\mathcal{T} \qquad O(3)_{orb} \qquad SU(2)_{spin}}^{\text{continuous}} \qquad\qquad \overbrace{P \qquad T}^{\text{discrete}}$$

spatial translation

rot^n in orbital space

rot^n in spin space

spatial inversion

time reversal

also (trivially) invariant under global "gauge transformation,

$$\Psi(r_1 r_2 \ldots r_N : \sigma_1\sigma_2\ldots\sigma_N) \rightarrow \exp i\varphi \cdot \Psi(r_1 .. r_N : \sigma_1 .. \sigma_N)$$

(2) QED Lagrangian (density):

$$\mathcal{L}(x) = -\frac{1}{4}\left(\frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}\right)^2 - mc^2\bar{\Psi}\Psi - \hbar c\left(\bar{\Psi}\gamma_\mu(\partial_\mu - ieA_\mu)\Psi\right)$$

invariant under:

Poincaré group (≡ Lorentz + space-time transl$^{n)}$,

C, P, T,

local gauge transf^n

$$\left(\Psi(x) \rightarrow e^{ie\theta(x)}\Psi(x), \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu\theta(x)\right)$$

BROKEN SYMMETRY (cont.)

H (or $\mathcal{L}$) is (exactly or approximately) invariant under all the operations of some symmetry group G. The thermodynamic equilibrium state is invariant only under the operations of some subgroup K∈G. (↑: K may be simply the identity!)

In principle, two cases:

(a) ∃ some small perturbation which is not invariant under (all operations of) G.

Examples:

CM: earth's magnetic field, lab bench. . . .

Particle theory: Coulomb interaction (in context of isotopic spin symmetry)

This is the (conceptually) "easy" case. In this case, ∃ same operation $\hat{R} \in G$ (but $\notin K$) s. t.

$\hat{R}|0\rangle \neq |0\rangle$ ← physically realized thermodynamic eq^{n} state

so in general $\exists\, \hat{\Omega}(x),\ \int \hat{\Omega}(x)dx = \Omega,$ s.t. $[\hat{\Omega}, \hat{R}] \neq 0$

"order parameter"

but nevertheless

$$\langle 0|\hat{\Omega}|0\rangle \neq 0.$$

(b) ∃ no small (physical) perturbation

Example (particle physics): Higgs mechanism.

This is the "difficult" case. For this, best df. is

$$\lim_{|\underset{\sim}{r}|\to\infty} \langle 0|\hat{\Omega}(\underset{\sim}{r})\,\hat{\Omega}(0)|0\rangle \neq 0$$

A SIMPLE EXAMPLE: HEISENBERG MAGNET

N quantum-mechanical spins on lattice:

$$\hat{H} = -J \underbrace{\sum_{i,m=n.n.}^{N} \hat{\underset{\sim}{S}}_i \cdot \hat{\underset{\sim}{S}}_j}_{} - \mu S_z \mathcal{H} \leftarrow \text{ext } \ell \text{ field} \parallel \hat{z}.$$

magnetic moment

invariant under 0(3)
[also $\mathcal{T}(\underset{\sim}{a})$]

Assume: $\mu\mathcal{H}/k_B T << 1.$

(a) J=0 ("ideal paramagnet")
spins independent, single-spin Zeeman energy competes with $k_B T$: e.g. for S=1/2.

$$P\uparrow / P\downarrow \sim \exp(\mu\mathcal{H}/kT)$$

$$\Rightarrow \langle S_z \rangle = \frac{1}{2} N \tanh(\mu\mathcal{H}/\mathrm{kT}) \approx \frac{1}{2} N . \frac{\mu\mathcal{H}}{kT} (= o(\mathcal{H}))$$

(b) J > 0 ("Heisenberg ferromagnet")
Entropy considerations favor "disorder" $(\langle S_z \rangle \to 0)$.
Interaction term: invariant under 0(3) $\Rightarrow$ favors spins lying $\parallel$, but does not specify common direction.
But, if (common) direction makes angle θ with z-axis, then

$$E_z \sim -N\mu\mathcal{H}\cos\theta$$

$$\Rightarrow \langle S_z \rangle \sim \frac{1}{2} N\mu \tanh(N\mu\mathcal{H}/kT)$$

Provided $N\mu\mathcal{H}/k_B T >> 1$,

$$\langle S_z \rangle \sim \frac{1}{2} N\mu, \; = \text{independent of } \mathcal{H}$$

In this example, $\hat{R} =$ (e.g.) rot^n around x-axis,

$$\hat{\Omega}(\underset{\sim}{r}) = S_z(\underset{\sim}{r}). \qquad \text{Note } [S_z, H] = 0$$

HEISENBERG MAGNET (cont.)

Recap: $$\hat{H} = -J \underbrace{\sum_{i,m=n.n.}^{N} \hat{\underset{\sim}{S}}_i \cdot \hat{\underset{\sim}{S}}_j}_{} \; (-\mu S_z \mathcal{H})$$

inv^t under $0(3), \Im_a, P, T$

(c) J<0 ("Heinsenberg antiferromagnet")

Interaction now favors nearest-neighbor spins lying antiparallel ("Néel state"):

$$\begin{array}{cccccc} \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \ldots \\ \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \end{array}$$

Uniform field $\mathcal{H}$ now does not "break" symmetry. What does?

(1) Generally, crystal-field effects break 0(3) symmetry (Heisenberg → Ising), but leave symmetry under T intact.

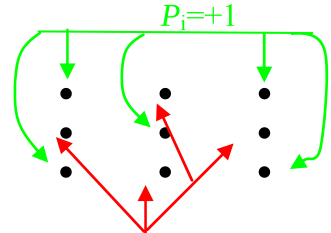
(2) V. weak inhomogeneous magnetic fields break residual T-invariance.

In this case, $\langle S_z \rangle = 0$. What is order parameter?

Ans: "Staggered" magnetization,

$$\underset{\sim}{N} \equiv \sum_i (-1)^{P_i} \langle \underset{\sim}{S}_i \rangle$$

$P_i \equiv$ "parity" of atomic site i



Note:

(a) in this case, $\left[\hat{\underset{\sim}{N}}, \hat{H} \right] \neq 0$

(b) $\underset{\sim}{N}$ is not invariant under T or $\Im_\ell$ alone, but is invariant under their combination.

Landau-Lifshitz Theory of Phase Trans^{NS}

(2nd or 1st order!)

Hamiltonian has some symmetry which is broken by formation of nonzero order parameter $\eta(x) \equiv \langle \Omega(x) \rangle$. At high enough T, expect entropy considerations to favor $\eta = 0$: at low T, interaction energy may favor finite η. Expand free energy FE-TS in powers of η: schematically.

$$F(T:\eta) = a_0(T) + a_1(T)0(\eta) + a_2(T)0(\eta^2) + a_3(T)0(\eta^3) + \ldots (+\text{gradient terms})$$

$a_1(T) \neq 0 \Rightarrow$ no phase transition

$a_1(T) \neq 0 \Rightarrow$ phase transition of 1st order (e.g. Xtal)

Fortunately, $F(T{:}\eta)$ must respect symmetry of $\hat{H}$! In particular, if $\hat{H}$ invariant under $\hat{\Omega}(x) \to -\hat{\Omega}(x)$, then all odd terms vanish. $\Rightarrow$ (poss. of) 2nd order phase transition.

Illustration: Heisenberg spins confined to plane: OP is complex scalar.

$$\eta(\underset{\sim}{r}) \equiv \left\langle S_x(\underset{\sim}{r}) + iS_y(\underset{\sim}{r}) \right\rangle$$

in this case, invariance requires

0(2) invariance

$$F\left\{T:\eta(\underset{\sim}{r})\right\} = \int dr\, \mathcal{F}(T:\eta(r))$$

$$\mathcal{F}(T:\eta(r)) = \mathcal{F}_0\,(T) + \propto(T)|\eta(r)|^2 + \frac{1}{2}\beta(T)|\eta(r)|^4 + \ldots$$

$$+\gamma(T)|\underset{\sim}{\nabla}\,\eta(r)|^2 + \ldots$$

0(2) invariance plus analyticity

cf: $\quad S = \int d^4x\, \mathcal{L}(x,t)$

$$\mathcal{L}(xt) \equiv \mathcal{L}_0 - m^2|\varphi(rt)|^2 - \left(\frac{\partial\varphi}{\partial x_\mu}\right)^2 - g|\varphi(rt)|^4$$

(φ^4 field theory)

LL THEORY OF (2^{ND} ORDER) PHASE TRANSITIONS (cont.)

Recap:

$$F\{T:\eta(\underset{\sim}{r})\} = \int d^3\underset{\sim}{r}\, \mathcal{F}(T:\eta(\underset{\sim}{r}))$$

$$\mathcal{F}=\mathcal{F}_0 + \alpha(T)|\eta(r)|^2 + \frac{1}{2}\beta(T)|\eta|^4 + \gamma(T)|\underset{\sim}{\nabla}\eta|^2 + \ldots$$

"Normal" state (η(r)=0) stable if $\alpha, \beta > 0$.

Typically, entropy S is decreasing f(ηand interaction en. is –const.|η|^2.

So, generically, in $F \equiv E - TS$,

$$\beta(T) > 0, \quad \gamma(T) > 0$$

$$\alpha(T) \sim -J_0|\eta|^2 + \zeta T|\eta|^2 \quad \left(\zeta \equiv \frac{\partial^2 S}{\partial \eta^2}|_{\eta=0}\right)$$

$$\equiv \alpha_0(T - T_c)|\eta|^2 \quad , \begin{cases} \alpha_0 \equiv \zeta, \\ T_c \equiv J_0/\zeta \end{cases}$$

So, approximately ($T \sim T_c$),

$$\mathcal{F}(\eta) = \alpha_0(T - T_c)|\eta|^2 + \frac{1}{2}\beta_0|\eta|^4 + \gamma_0|\underset{\sim}{\nabla}\eta|^2 \; (-\eta\mathcal{H}_{ext})$$

(canonical form of LL("mean-field") free energy)

Consequences:

(1) For $T>T_c$, η=0

(2) For $T<T_c$, $\eta = \sqrt{\frac{\alpha_0}{\beta_0}}(T_c - T)^{1/2}$

<u>phase</u> of η det^d by $\mathcal{H}_{ext}$

(3) Deformation ("twist") of $\eta(\underset{\sim}{r})$ const en. $\sim |(\nabla\eta)|^2$

$\Rightarrow$ Goldstone boson ($\omega \to 0$ for $\underset{\sim}{k} \to 0$) (generic for broken <u>continuous</u> symmetry)

↑: spin + structural glasses
dissociation
FQH ("topologically ordered" state)
etc.

Symmetry Breaking in Field Theory

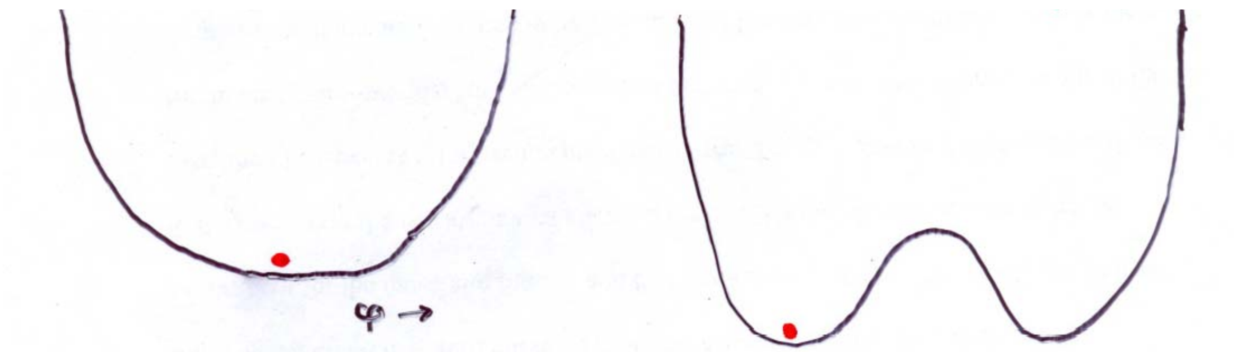
2nd order phase trans^{n}:

$$\mathcal{F}\left[\eta(r)\right]=\left(\mathcal{F}_0+\right)\alpha(T)\,|\eta(r)|^2+\frac{1}{2}\beta_0\,|\eta(r)|^4+\gamma_0\,|\underset{\sim}{\nabla}\,\eta(r)|^2$$

φ^4 field theory:

$$\mathcal{H}\left[\varphi(rt)\right] = \left(\left(\frac{\partial\varphi}{\partial t}\right)^2+\right) m^2\,|\varphi(rt)|^2+|\frac{1}{2}g\varphi|^4+|\underset{\sim}{\nabla}\,\varphi(rt)|^2$$

No gauge coupling
(so far)



$A.\ \propto(T)\,(m^2)>0$ $\qquad$ $B.\ \propto(T)\,(m^2)<0$

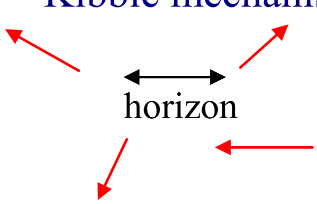
"Disordered" phase: $\langle\varphi(rt)\rangle=0$

"Ordered" phase:

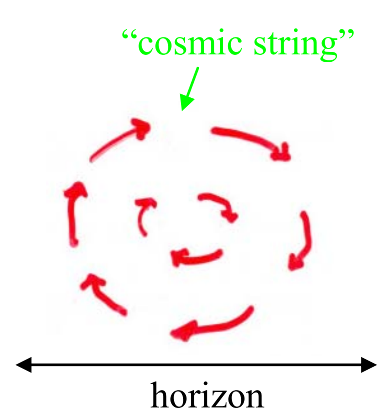
$$\left\langle\varphi(\underset{\sim}{r}t)\right\rangle\neq 0 \qquad \text{or} \qquad \left\langle\varphi(\underset{\sim}{r}t)\,\varphi(\underset{\sim}{r}t)\right\rangle+0,|\underline{r}-\underline{r}'|\to\infty.$$

Some consequences:

1. Kibble mechanism



$\Rightarrow$



2. HIGGS MECHANISM

Ex: massless vector boson with gauge coupling to scalar field ("toy model" of EW theory)

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m^2\varphi^2 - \frac{1}{2}g\varphi^4 + j_\mu A^\mu$$

Current of φ

$\Rightarrow$ (Coulomb gauge)

$$\mathcal{H} = \frac{1}{2}\left\{\left(\frac{\partial \underset{\sim}{A}}{\partial t}\right)^2 + \left(\frac{\partial \varphi}{\partial t}\right)^2\right\} + m^2\varphi^2 + \frac{1}{2}g\varphi^4 + (\text{curl } \underset{\sim}{A})^2 + |(\underset{\sim}{\nabla} - ie\underset{\sim}{A})\varphi|^2$$

Groundstate:

(a) $m^2 > 0$: no $SB, \langle\varphi\rangle = 0$

$\Rightarrow$ eff Hamiltonian for $\underset{\sim}{A}$ just $\frac{1}{2}\left\{(\partial A/\partial t)^2 + (\text{curl } \underset{\sim}{A})^2\right\}$

$\Rightarrow$ vector boson remains massless (ω=ck)

(b) $m^2 < 0$: SB,

$$\langle\varphi(\underset{\sim}{r})\rangle \equiv \varphi_0 \neq 0$$

(i) $e = 0$, dynamics of $\underset{\sim}{A}$ unaffected.

(ii) $e \neq 0$: Can $\langle\varphi\rangle$ "follow" $\underset{\sim}{A}$ to make KE term 0?

This would require: $\left(\varphi \equiv |\varphi| e^{i\chi}\right)$

$$\underset{\sim}{\nabla}\chi(rt) = e\underset{\sim}{A}(rt)$$

But, if curl $\underset{\sim}{A} \neq 0$ ('transverse" excit^n) this is not possible!

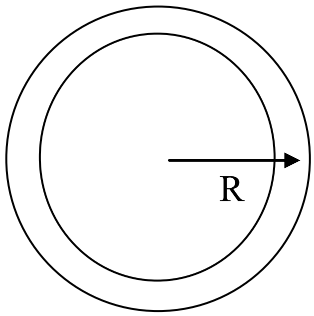
$\Rightarrow$ φ stays at original value φ_0, and eff. Hamiltonian for $\underset{\sim}{A}$ becomes

$$\mathcal{H}(\underset{\sim}{A}) = \frac{1}{2}\left\{\left(\frac{\partial A}{\partial t}\right)^2 + (\text{curl } \underset{\sim}{A})^2 + e^2\varphi_0^2\underset{\sim}{A}^2\right\}$$

$\Rightarrow$ Vector boson acquires mass $\left(m_A = e|\varphi_0|\right)$

<u>SUPERCONDUCTIVITY AND SUPERFLUIDITY:</u>
<u>SPONTANEOUSLY BROKEN U(1) GAUGE SYMMETRY?</u>

Phenomenology:



df. $\omega_c \equiv \hbar / mR^2$
$\equiv$ "quantum unit of angular velocity"

	<u>Stable</u>	<u>Metastable</u>
A. Superconductors	diamagnetism $\Rightarrow$ Meissner effect	persistent currents
B. Superfluid ^{4}He	NCRI (Hess-Fairbank effect)	〃

Bohr-v. Leeuwen theorem (+ neutral analog) $\Rightarrow$ neither diamagnetism nor NCRI possible in classical system

Usual account of Meissner effect:

$$\hat{H} = \frac{\hbar^2}{2m}\int |(\nabla\Psi)|^2 \, dr + \frac{1}{2}\int V(\underset{\sim}{r} - \underset{\sim}{r})\Psi^+(\underset{\sim}{r})\Psi^+\left(\underset{\sim}{r}'\right)\Psi\left(\underset{\sim}{r}'\right)\Psi\left(\underset{\sim}{r}\right) d\underset{\sim}{r} \; d\underset{\sim}{r}'$$

Invariant under global U(1) transf^{n}

$$\Psi(\underset{\sim}{r}) \to e^{i\alpha}\,\Psi(\underset{\sim}{r}) \qquad (\alpha = const.)$$

in "super" state, U(1) symm. spontaneously broken,

$$\left\langle \Psi(\underset{\sim}{r}) \right\rangle \neq 0 \qquad \Big(\text{or} \left\langle \Psi(r)\Psi(r') \right\rangle \underset{|\underset{\sim}{r} - \underset{\sim}{r}'| \to \infty}{\neq} 0.\Big)$$

CONVENTIONAL ACCOUNT OF MEISSNER EFFECT (cont.)

SB of U(1) symm. $\Rightarrow \langle \Psi(\underset{\sim}{r}) \rangle \neq 0$.

With gauge coupling to (real) EM field,

$$KE = \frac{\hbar^2}{2m} \int |\left(\underset{\sim}{\nabla} - ie\underset{\sim}{A}\right)\Psi|^2 \, d\underset{\sim}{r}$$

$\Rightarrow$ "Higgs-like" mechanism, photon acquires mass. (in sup^r)

$\Rightarrow$ EM field falls off exp'ly in interior of superconductor (Meissner effect)

WHAT'S WRONG WITH THIS PICTURE?

Well, not exactly wrong, but in CM context:

(1) "Longitudinal-transverse" symmetry (from Lorentz invariance) lost in CM problems.

(2) $\langle \psi(\underset{\sim}{r}) \rangle \neq 0 \Rightarrow \Psi = \sum_N c_N \psi$ ←violates superselection rule on N

(but alt^n d.f., $\langle \psi^+(\underset{\sim}{r})\,\psi(\underset{\sim}{r}) \rangle \not\to 0$ for $|\underset{\sim}{r} - \underset{\sim}{r}'| \to \infty$, is OK)

(3) SB U(1) symmetry is sufficient for Meissner effect (or NCRI). It is not sufficient for the stability of supercurrents.

(4) SB U(1) symmetry is not necessary for stability of supercurrents, and poss not even for NCRI.

Alternative Approach to Superconductivity and Superfluidity
(Penrose-Onsager, BCS, Yang…)

A. Free Bose gas (ultra-naïve model for liquid ^{4}He)
N conserved bosons in free space, in thermal eqn.

$$\langle n_k(T)\rangle = (\exp\beta(\underset{\substack{\uparrow\\ \text{chem. pot}^\ell,\ \leq 0}}{\epsilon_k - \mu}) - 1)^{-1} \qquad (\beta \equiv 1/k_B T)$$

$$\Rightarrow \sum_{k\neq 0} \langle n_k(T:\mu)\rangle \leq \sum_{k\neq 0} \langle n_k(T:\mu=0)\rangle \equiv N_{exc}(T)$$

If $N_{exc}(T) < N$, then Bose-Einstein condensation (BEC)

$$\underset{\substack{\uparrow\\ \underset{\sim}{k}=0 \text{ state}}}{N_0}(T) = N - N_{exc}(T) = 0(N)$$

Note: effect not of interactions but of "levelling of entropic playing field"

B. Interacting Bose gas (not necessarily in eq$^{\text{m}}$) (but pure state, for moment)
(Penrose, Onsager)

single-particle density matrix

$$\text{Df: } \rho_1(\underset{\sim}{r},\underset{\sim}{r}':t) \equiv \int dr_2..dr_N \Psi_N^*(\underset{\sim}{r}, r_2...r_N)\Psi_N(\underset{\sim}{r}', \underset{\sim}{r}_2...r_N)$$

At any given t:

orthonormal set

$$\rho_1(\underset{\sim}{r},\underset{\sim}{r}':t) - \sum_i n_i(t)\chi_i^*(\underset{\sim}{r}:t)\chi_i(\underset{\sim}{r}':t)$$

≈ "av. no. in state i at time t"

("simple") BEC iff one and only one state ("0") s. t.

$$n_0(t) = 0(N), \quad n_i(t) = 0(1) \text{ for } i \neq 0$$

(otherwise, normal (all $n_i 0(1)$) or poss. "fragmented")

Penrose-Onsager df. of order parameter (OP)

Assume (simple) BEC in state $\chi_0(\underset{\sim}{r}:t)$,

Not nec[y]? $\underset{\sim}{k}=0$ state!

then $(n_0 \to N_0)$

$$\Psi(\underset{\sim}{r},t) \equiv \sqrt{N_0(t)}\,\chi_0(\underset{\sim}{r}:t)$$

"condensate no." "condensate w.f."

Note: From its def[n], global phase of $\Psi(rt)$ (like that of single-particle Schrödinger w.f.) is physically meaningless.

Does BEC always occur in an interesting system of bosons at low enough T?

No! (counterexx: solid ^{4}He, 2D, KSA state…

C. Interacting Fermi system (not nec[y] in eq[m]) (BCS, Yang)

Df: 2-particle density matrix

$$\rho(\underset{\sim}{r}_1\,\underset{\sim}{r}_2\,\sigma_1\,\sigma_2,\underset{\sim}{r}_1'\,\underset{\sim}{r}_2'\,\sigma_1'\,\sigma_2'\,t) \equiv \sum_{\sigma_3..\sigma_N}\int dr_3..dr_N$$

$$\Psi_N^*(r_1\,\sigma_1,r_2\,\sigma_2,r_3\,\sigma_3\ldots r_N\sigma_N:t)\,\Psi_N\left(r_1'\sigma_1'r_2'\sigma_2',r_3\sigma_3..r_N\sigma_N:t\right)$$

At any given t:

orthonormal set

$$\rho(r_1r_2\,\sigma_1\,\sigma_2,r_1'\,r_2'\sigma_1'\sigma_2':t)=\sum_i n_i(t)\chi_i^*(r_1r_2\sigma_1\sigma_2:t)\,\chi_i(r_1'\,r_2'\sigma_1'\sigma_2':t)$$

"av. no. of pairs in 2p state i at time t"

(simple) Cooper pairing ("pseudo-BEC") iff for one and only one value of i,

$$n_0(T)=0(N),\; n_i(t)=0(1)\quad i\neq 0.$$

(otherwise normal or fragmented)

BCS-Yang df. of OP in Fermi system:

Assume (simple) Cooper pairing in state χ_0, then

$$\Psi(r_1 r_2, \sigma_1 \sigma_2 : t) \equiv \sqrt{N_0(t)}\, \chi_0 (r_1 r_2, \sigma_1 \sigma_2 : t)$$

“condensate no.” “condensate w.f.”

Note:

(a) $N_0(t) = 0(N)$ *not* $0(N^2)$! (Yang)

(b) For noninteracting F. system, $N_0(t) \equiv 0$

(c) As in Bose case, global phase of χ_0 physically meaningless

(d) In general, $\chi_0(\underset{\sim}{r}_1 \underset{\sim}{r}_2 \sigma_1 \sigma_2 : t)$ (and hence Ψ) has nontrivial <u>internal structure</u>:

$$\Psi(r_1 r_2, \sigma_1 \sigma_2 : t) \cong \Psi(\underset{\sim}{R}, t)\, f_{R}(\underset{\sim}{\rho}, \sigma_1 \sigma_2 : t)$$

For simple s-wave pairing (BCS theory) $f(p_1 \sigma_1 \sigma_2 : t)$ is fixed by energetics:

$$f = 2^{-1/2} (\uparrow_1 \downarrow_2 - \downarrow_1 \uparrow_2) \cdot f\left(|\underset{\sim}{\rho}|\right)$$

Spin singlet s-wave

Then $\Psi(\underset{\sim}{R}, t)$ is “macroscopic wave function” of Ginzburg and Landau, i.e.

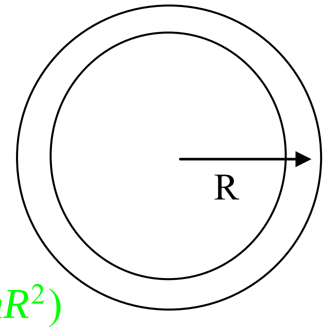
“order parameter” in BCS superconductor is COM wave function of Cooper pairs.

EXPLANATION OF SUPERFLUIDITY IN TERMS OF BEC

A. Hess-Fairbank effect (NCRI)

When container rotates with ang. velocity $\underset{\sim}{\omega}$, correct quantity to minimize is

$$F_{eff} = F - \underset{\sim}{\omega} \cdot \langle \underset{\sim}{L} \rangle \equiv \underbrace{H - \underset{\sim}{\omega} \cdot \langle L \rangle}_{H_{eff}} - TS$$



ℓ ↑

Ex: from Bose gas.

ang. mom^m q. no.

$\ell = \text{integer}, E_{eff}(\ell) = \ell^2 \hbar \omega_c - \ell \hbar \omega \qquad (\omega_c \equiv \hbar / mR^2)$

(typically $\sim 10^{-2} - 10^{-4} s^{-1}$)

(a) T »T$_c$:

$$\langle n_\ell(T) \rangle = \text{const. } \exp - \beta E_{eff}(\ell)$$

slowly varying $f(\ell)$ $\qquad \langle \ell^2 \rangle \sim \dfrac{kT}{\hbar \omega_c} \gtrsim 10^{14}$

$$\Rightarrow L = \sum_\ell \ell \hbar \langle n_\ell(T) \rangle \cong mR^2 \omega = I_{c\ell} \omega.$$

$\Rightarrow$system rotates with container

$L / I_{c\ell}$ ↑

(b) $T < T_c$. $\left(N_0(T) = 0(N) \right)$

ω_c

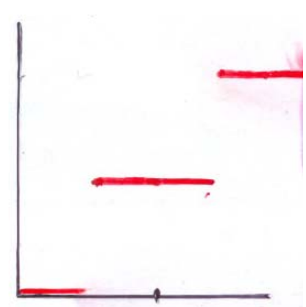
$\omega \rightarrow$

Condensate must form in unique state with lowest $E_{eff}(\ell)$.

This corresponds to $\ell = \text{int}\left(\omega / \omega_c - 1/2 \right) \equiv \ell_0)$

Contribution of condensate to ang. momentum is simply $N_0(T) \ell_0 \hbar$, so at $T = 0$:

$L / I_{e\ell}$



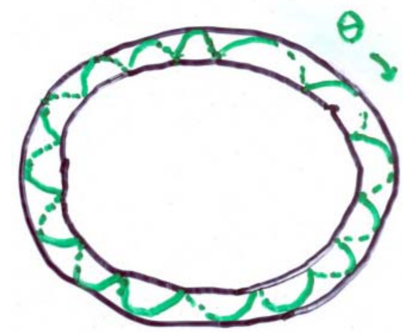
Considerations for interacting Bose system qualitatively similar.

ω_c

↑

arrange slope 1

B. Metastability of supercurrents



System initially rotated with int. $(\omega/\omega_c - 1/2) = \ell_0$, cooled through T_c while rotating: condensate forms in state with angular momentum $\ell_0 (\neq 0)$. When rotation is stopped, $E_{eff}(\ell) = \ell^2 \hbar\omega_c \Rightarrow$ stable state is $\ell = 0$, actual state metastable. Why no decay?

$$\underbrace{\psi_i = \exp i\ell_0\theta}_{\text{“winding no.”}} \qquad \underbrace{\psi_f = \text{const.}}_{\text{winding no. }=0}$$

Electron in atom (semiclassical approximation):
More generally, density $= |a|^2 + |b|^2 + \underbrace{\text{Re } 2a^* b \cos \ell_0 \theta}_{\text{inhomogeneous}}$

For e⁻ in atom $(E \propto |\psi|^2)$ inhomogeneity averages to zero in E.
For an interacting Bose system,

$\exists$ term in energy $\propto |\psi|^4$.

$\Rightarrow$ inhomogeneous states energetically disfavored
$\Rightarrow$ “topological” conservation of winding no.
$\Rightarrow$ supercurrent metastable

Similar arguments for charged system
(supercurrent metastability:
NCRI $\rightleftarrows$ diamagnetism $\Rightarrow$ Meissner effect)
Fermi system: argts. similar but $\Psi(rt)$ now 2-particle function (so. e.g. $\omega_c = (\hbar/2m)R^{-2}$)

SPONTANEOUSLY BROKEN U(1) GAUGE SYMMETRY (and BEC): SOME PROBLEMS

Formally, the OP as defined from BEC can be obtained equivalently from symmetry-breaking: in particular, can define (e.g. in Bose system) ODLRO by

$$P_1(rr') \equiv \langle \psi^+(\underset{\sim}{r})\psi(\underset{\sim}{r}') \rangle \underset{|\underline{r}-\underline{r}'|\to\infty}{\longrightarrow} \chi_0^*(r)\chi_0(r')$$

$$\Rightarrow \text{"}\langle \psi(r) \rangle\text{"} \equiv \chi_0(r)$$

But, explicit df via BEC focuses on some interesting issues:

(1) Why does BEC happen?

(2) Is BEC sufficient for "superfluidity"?
In sense of NCRI, yes
In sense of metastable supercurrents, no: we require in addition
(i) Net repulsive interaction ($a|\psi|^4$, $a > 0$)
(ii) order parameter a complex order

(3) Is BEC necessary for "superfluidity"?
In sense of persistent supercurrents, no (quasi-1 D systems)
In sense of NCRI, ?? ("Japanese-bus" situations?)

(4) More generally, malign effects of thermal averaging

(5) Conceptual problems with Kibble mechanism
(cf. optical-lattice expts)

MORE SOPHISTICATED SUPERFLUID/SUPERCONDUDTOR OP'S

So far, in considering Fermi systems with Cooper pairing, assumed dependence of OP

$$\Psi(\underset{\sim}{R}, \underset{\sim}{\rho}, \sigma_1\sigma_2)$$

on $\underset{\sim}{\rho}\sigma_1\sigma_2$ corresponds to singlet s-wave and hence is fixed by energetics. But in e.g. superfluid ^{3}He, OP is triplet and p-wave: e.g. in A phase,

COM at rest

$$\Psi(\underset{\sim}{\rho} : \sigma_1\sigma_2) \sim f(|\rho|) \sin\theta\, e^{i\varphi} \cdot \frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 + \downarrow_1\uparrow_2)$$

$\ell = 1$ with same (orbital) axis $\underset{\sim}{\ell}$

S=1, S_z=0 with same (spin) axis $\underset{\sim}{d}$

To the extent that $\hat{H}$ is invariant under rot^n of orbital and spin axes <u>separately</u>, $O(3)_{orb}$ and $SU(2)_{spin}$ still unbroken: will be broken by walls, margnetic field, etc.

What if $\hat{H}$ invariant under <u>total</u> rot^n but not under <u>relative</u> rot^n of spin and orbital coords? (e.g. nuclear dipole interaction). Then thermodynamic eqn. state may have similar props:
e.g. ^{3}He-B,

start with ^3P_o configuration, rotate through "spin-orbit rot^n $\angle$" ($\cong 104^\circ$) around <u>orb</u>. axis $\underset{\sim}{\omega}$

$\Rightarrow$ very anomalous NMR props.

Cf. also: cuprate superconductors ($d_{x^2-y^2}$)

Sr_2Ru0_4 (p-wave, T-violating) (Kidwiringa et al.)

BACK TO "MEANING" OF OP

"Absolute" phase of OP is meaningless. What about relative phase?

Ex: ^{3}He-A: with appropriate choice of axes, usual form of OP is

$$\Psi\left(\rho,\sigma_1\sigma_2\right)\sim f\left(\underset{\sim}{\rho}\right)\left(\frac{1}{\sqrt{2}}\left(\uparrow\uparrow+e^{i\Delta\varphi}\downarrow\downarrow\right)\right)$$

But: corresponding MBWF is not eigenf'n of S_z
$\Rightarrow$ energetically disadvantageous?

$$\left[S_z,\Delta\varphi\right]=2i$$
$$E\left(S_z,\Delta\varphi\right)=\frac{1}{2}\frac{S_z^2}{\chi}-g_D\cos\Delta\varphi$$

polarization

Nuclear dipole

Crux: both χ and $g_D \propto N$! Hence for $N\to\infty$,
$\Delta\varphi$ well-defined, at expense of S_z

But for $N\to 0$ (or $g_D\to 0$), S_z well-df, $\Delta\varphi$ undefined
(cf. ultra-small capacitance Josephson junctions)

Even if $\Delta\varphi$ "initially" undefined, may be defined "by measurement" (cf. MIT expt. on ^{87}Rb) relevant to Kibble mechanism/optical-lattice expts?

BOTTOM LINE: QUITE A BIT REMAINS MURKY!