MAJORANA FERMIONS IN CONDENSED-MATTER PHYSICS

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based in part on joint work with Yiruo Lin

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Abdus Salam's 90th birthday

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Reminder:

Majorana fermions (M.F.'s) in particle physics (E. Majorana, 1937)

An M.F. is a fermionic particle which is its own antiparticle:

$$\psi(\boldsymbol{r}) \equiv \psi^\dagger(\boldsymbol{r}) \Rightarrow \{\psi(\boldsymbol{r}), \psi(\boldsymbol{r}')\} = \delta(\boldsymbol{r} - \boldsymbol{r}')$$

How can such a particle arise in condensed matter physics?
Example: "topological superconductor".

Mean-field theory of superconductivity (general case):
Consider general Hamiltonian of form $\hat{H}_0 + \hat{V}$ where

$$\hat{H}_0 \equiv \frac{\hbar^2}{2m} \int \boldsymbol{\nabla}\psi^\dagger(\boldsymbol{r}) \cdot \boldsymbol{\nabla}\psi(\boldsymbol{r}) d\boldsymbol{r} + \sum_{\alpha\beta} U_{\alpha\beta}(\boldsymbol{r})\, \psi_\alpha^\dagger(\boldsymbol{r})\psi_\beta(r)$$

$$\hat{V} \equiv \frac{1}{2} \sum_{\alpha\beta\gamma\delta} \iint d\boldsymbol{r}\, d\boldsymbol{r}' V_{\alpha\beta\gamma\delta}(\boldsymbol{r}, \boldsymbol{r}')\psi_\alpha^\dagger(r)\psi_\beta^\dagger(r')\psi_\gamma(r')\psi_\delta(r)$$

"SBU(1)S"

Introduce notion of spontaneously broken U(1) symmetry ⇒ particle number not conserved ⇒ (even-parity) GS of form

$$\Psi_{(\text{even})} = \sum_{\substack{N \\ =\text{even}}} C_N \Psi_N$$

⇒ quantities such as $\langle \psi_\alpha(r)\psi_\beta(r) \rangle$ can legitimately be nonzero.

Thus

mean-field

$\hat{H} \rightarrow \hat{H}_0 + \hat{V}_{mf}$ where (apart from Hartree-Fock terms)

operator

$$\hat{V}_{mf} \equiv \sum_{\alpha\beta} \iint dr\, dr' \Delta_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}') \underbrace{\psi_\alpha^\dagger(r)\psi_\beta^\dagger(r')}_{} + H.C.$$

$$\Delta_{\alpha\beta}(r, r') \equiv \sum_{\gamma\delta} V_{\alpha\beta\gamma\delta}(r, r') \langle \psi_\gamma(r')\psi_\delta(r) \rangle \quad \leftarrow \quad \text{c-number}$$

[in BCS case, reduces to

$$\hat{V}_{mf} = \int dr\, \Delta(r)\psi_\uparrow^\dagger(r)\psi_\downarrow^\dagger(r) + H.C., \;\; \Delta(r) \equiv V_0 \langle \psi_\downarrow(r)\psi_\uparrow(r) \rangle]$$

↙ Bogoliubov-de Gennes

Thus, mean-field (BdG) Hamiltonian is schematically of form

$$\hat{H}_{mf} =$$

$$\sum_{\alpha\beta}\left\{\underbrace{\int dr\, K_{\alpha\beta}(r)\psi_\alpha^\dagger(r)\psi_\beta(r) + \frac{1}{2}\iint dr\, dr'\Delta_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}')\psi_\alpha^\dagger(r)\psi_\beta^\dagger(r')}_{\text{bilinear in } \psi_\alpha(r),\, \psi_\alpha^\dagger(r):} + HC\right\}$$

(with a term $\mu\delta_{\alpha\beta}$ included in $K_{\alpha\beta}(r)$ to fix average particle number $\langle\hat{N}\rangle$.) $\hat{H}_{mf}$ does not conserve particle number, but does conserve particle number parity, so consider even parity. (Then can minimize $\hat{H}_{mf}$ to find even-parity CS, but) in our context, interesting problem is to find simplest fermionic (odd-parity) states ("Bogoliubov quasiparticles"). For this purpose write schematically (ignoring (real) spin degree of freedom)

$$\hat{\gamma}_i^\dagger = \int \{u_i(r)\hat{\psi}^\dagger(r) + v_i(r)\hat{\psi}(r)\}\, dr \qquad \left(\equiv \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}\right)$$ ← "Nambu spinor"

and determine the coefficients $u_i(r), v_i(r)$ by solving the Bogoliubov-de Gennes equations

$$[\hat{H}_{mf}, \hat{\gamma}_i^\dagger] = E_i\hat{\gamma}_i^\dagger$$

so that

$$\hat{H}_{mf} = \sum_i E_i\gamma_i^\dagger\gamma_i + \text{const.}$$

(All this is standard textbook stuff…)

Note crucial point: In mean-field treatment, fermionic quasiparticles are quantum superpositions of particle and hole ⇒ do not correspond to definite particle number (justified by appeal to SBU(1)S). This “particle-hole mixing” is sometimes (misleadingly) regarded as analogous to the mixing of different bands in an insulator by spin-orbit coupling. (hence, analogy “topological insulator” ⇄ topological superconductor.)

Majoranas

Recap: fermionic (Bogoliubov) quasiparticles created by operators

$$\gamma_i^\dagger = \int dr \left\{u_i(r)\psi^\dagger(r) + v_i(r)\psi(r)\right\}$$

with the coefficients $u_i(r), v_i(r)$ given by solution of the BdG equations

$$\left[\hat{H}_{mf}, \gamma_i^\dagger\right] = E_i\gamma_i$$

Question: Do there exist solutions of the BdG equations such that
$\gamma_i^\dagger = \gamma_i$ (and thus $E_i = 0$)?

This requires (at least)

1. Spin structure of $u(r), v(r)$ the same ⇒ pairing of parallel spins (spinless or spin triplet, not BCS s-wave)

2. $u(r) = v^*(r)$

3. "interesting" structure of $\Delta_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}')$, e.g. "$p + ip$" $\left(\Delta(\boldsymbol{r}, \boldsymbol{r}') \equiv \Delta(\boldsymbol{R}, p) \sim \Delta(R)\left(p_x + ip_y\right)\right)$

Case of particular interest: "half-quantum vortices" (HQV's) in Sr_2RuO_4 (widely believed to be $(p+ip)$ superconductor). In this case a M.F. predicted to occur in (say) ↑↑ component, (which sustains vortex), not in ↓↓ (which does not). Not that vortices always come in pairs (or second MF solution exists on boundary)

Why the special interest for topological quantum computing?

(1) Because MF is exactly equal superposition of particle and hole, it should be undetectable by any local probe.

(2) MF's should behave under braiding as Ising anyons*:
if 2 HQV's, each carrying a M.F., interchanged, phase of MBWF changed by $\pi/2$ (note not π as for real fermions!)

So in principle‡:

(1) create pairs of HQV's with and without MF's

(2) braid adiabatically

(3) recombine and "measure" result

⇓

(partially) topologically protected quantum computer!

* D. A. Ivanov, PRL **86**, 268 (2001)
‡ Stone & Chung, Phys. Rev. B **73**, 014505 (2006)

Comments on Majarama fermions (within the standard "mean-field"approach)

(1) What is a M.F. anyway?

Recall: it has energy exactly zero, that is its creation operator $\gamma_i^\dagger$ satisfies the equation

$$[H, \gamma_i^\dagger] = 0$$

But this equation has two possible interpretations:

(a) $\gamma_i^\dagger$ creates a fermionic quasiparticle with exactly zero energy (i.e. the odd- and even-number-parity GS's are exactly degenerate)

(b) $\gamma_i^\dagger$ annihilates the (even-parity) groundstate ("pure annihilator")

However, it is easy to show that in neither case do we have $\gamma_i^\dagger = \gamma_i$. To get this we must superpose the cases (a) and (b), i.e.

a Majarana fermion is simply a quantum superposition of a real Bogoliubov quasiparticle and a pure annihilator.

But Majorana solutions always come in pairs ⇒ by superposing two MF's we can make a real zero-energy fermionic quasiparticle

HQV1 HQV2

$\gamma_1^\dagger$ $\gamma_2^\dagger$

Bog. qp. ⟶ $\alpha^\dagger \equiv \gamma_1^\dagger + i\gamma_2^\dagger$

The curious point: the extra fermion is "split" between two regions which may be arbitrarily far apart! (hence, usefulness for TQC)

Thus, e.g. interchange of 2 vortices each carrying an MF ~ rotation of zero-energy fermion by π. (note predicted behavior (phase change of π/2) is "average" of usual symmetric (0) and antisymmetric (π) states)

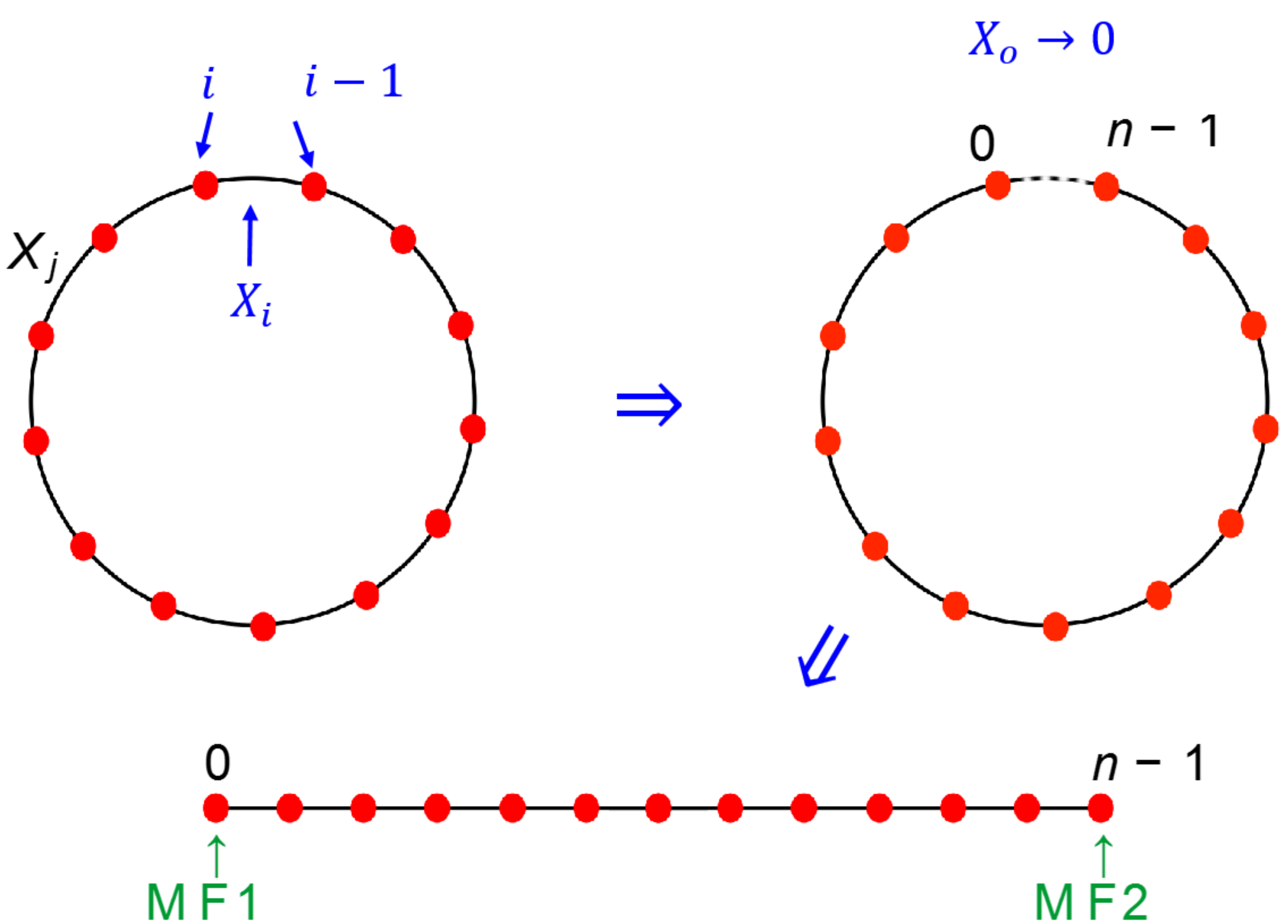
An intuitive way of generating MF's in the KQW:

Kitaev quantum wire

For this problem, fermionic excitations have form

$$\alpha_i^\dagger = (a_i^\dagger + i a_i) + (a_{i-1}^\dagger + i a_{i-1})$$

so localized on links not sites. Energy for link $(i, i-1)$ is X_i



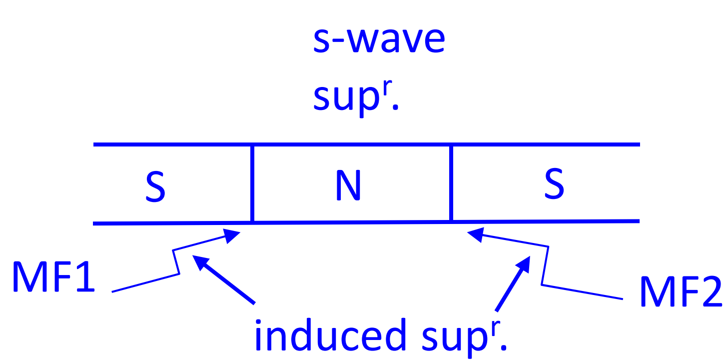
Comments on M.F.'s (within standard mean-field approach) (cont.)

(2) The experimental situation

Sr_2RuO_4: so far, evidence for HQV's, none for MF's.

^{3}He-B: circumstantial evidence from ultrasound attenuation

Alternative proposed setup (very schematic)



zero-bias anomaly

Detection: ZBA in I-V characteristics

(Mourik et al., 2012, and several subsequent experiments)

dependence on magnetic field, s-wave gap, temperature... roughly right

"What else could it be?"

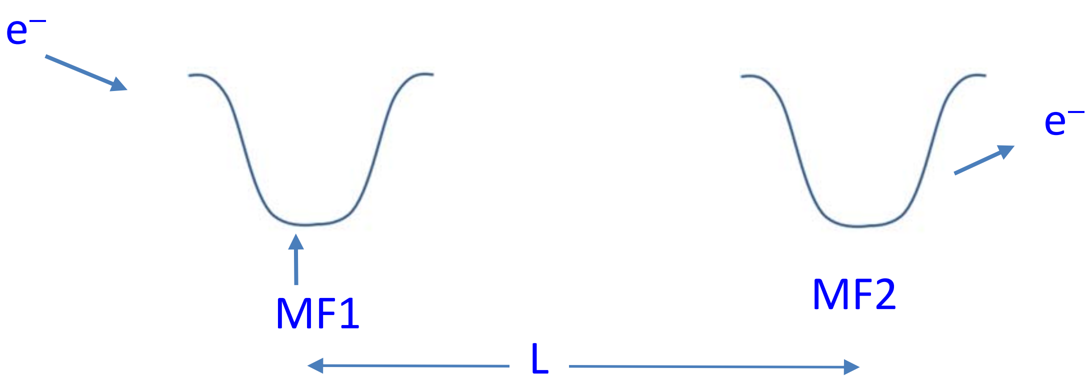
Answer: quite a few things!

Second possibility: Josephson circuit involving induced (p-wave-like) sup^y.

Theoretical prediction: "4π-periodicity" in current-phase relation.

Problem: parasitic one-particle effects can mimic.

One possible smoking gun: teleportation!



$\Delta T \ll L/v_F$? ← Fermi velocity

Problem: theorists can't agree on whether teleportation is for real!

Majorana fermions: beyond the mean-field approach

Problem: The whole apparatus of mean-field theory rests fundamentally on the notion of SBU(1)S ← spontaneously broken U(1) gauge symmetry:

$$\Psi_{\text{even}} \sim \sum_{\substack{N= \\ \text{even}}} C_N \, \Psi_N \qquad \left(C_N \sim |C_N| e^{iN\varphi}\right)$$

$$\Psi_{\text{odd}}^{(c)} \sim \int dr \left\{ u(r) \hat{\psi}^\dagger(r) + v(r) \hat{\psi}(r) \right\} |\Psi_{\text{even}}\rangle \;\; \left(\equiv \hat{\gamma}_i^\dagger |\Psi_{\text{even}}\rangle\right)^*$$

But in real life condensed-matter physics,

SB U(1)S IS A MYTH!!

This doesn't matter for the even-parity GS, because of "Anderson trick":

$$\Psi_{2\text{N}} \sim \int \Psi_{\text{even}}(\varphi) \exp -iN\varphi \, d\varphi$$

But for odd-parity states equation (*) is fatal! Examples:

(1) Galilean invariance

(2) NMR of surface MF in ^{3}He-B

We must replace (*) by

creates extra Cooper pairs

$$\hat{\gamma}_i^\dagger = \int dr \left\{ u(r)\hat{\psi}^\dagger(r) + v(r)\hat{\psi} C^\dagger \right\}$$

This doesn't matter, so long as Cooper pairs have no "interesting" properties (momentum, angular momentum, partial localization...)

But to generate MF's, pairs must have "interesting" properties!

⇒ doesn't change arguments about existence of MF's, but completely changes arguments about their braiding, undetectability etc.

Need completely new approach!