COOPER PAIRING IN "EXOTIC" FERMI SUPERFLUIDS: AN ALTERNATIVE APPROACH

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The established wisdom*:

BCS theory (including “spontaneously broken U(1) symmetry”)

↓

“topological superconductors” (e.g. $Sr_2RuO_4,\ {}^3He-A$)

↓

Majorana fermions: (nonabelian statistics)

↓

(Ising) topological quantum computation (and other exotica)

The $64K question:

Is the established wisdom correct?

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* E.g. Read & Green, Phys. Rev. B **61**, 10267 (2000)

Ivanov, Phys. Rev. **86**, 268 (2001)

Stern, von Oppen & Mariani, Phys. Rev. B **70**, 205338 (2004

Chung & Stone, J. Phys. A **40**, 4923 (2007)

Nayak, et al. Rev. Mod. Phys. **80**, 1083 (2008)

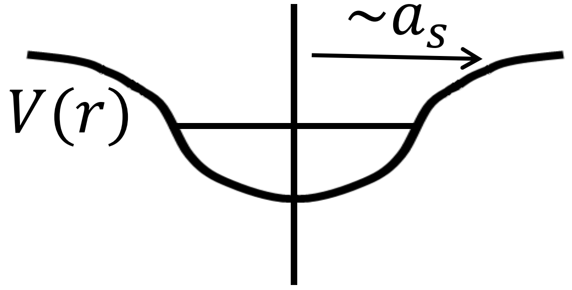
Read, Phys. Rev. B **79**, 045308 (2009)

Basic description of Cooper pairing ($T = 0$)

Illustrative example: N(=even) spin $-1/2$ fermions, mass m, in volume Ω, interacting via isotropic attractive potential $V(r)$ of range $r_o \ll (\Omega/N)^{1/3}$ but adjustable strength. (example: ultracold Fermi gas, Feshbach resonance). Effect of potential encapsulated in s-wave scattering length a_s.

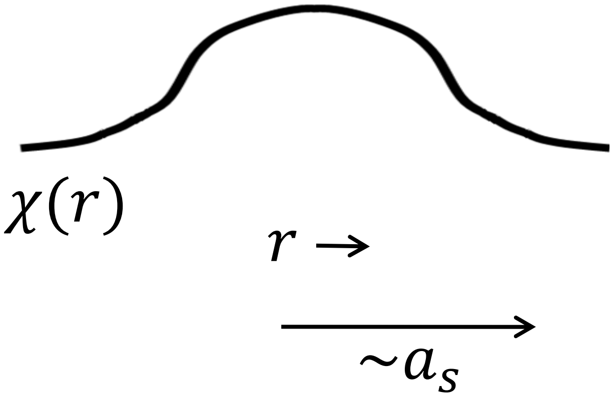
$\equiv r_{int}$

1. "BEC limit" (strong attraction)



2 fermions form simple s-wave diatomic molecule, with radius $\lesssim a_s (> 0) \ll r_{int}$, binding energy $\sim \hbar^2/ma_s^2\,, rl^{vc}$. w.f. $\chi(r)$ Since $a_s \ll r_{int}$, molecules do not overlap $\Rightarrow$ can be regarded as $N/2$ simple bosons, with (com) coords $\boldsymbol{R}_i$, small residual interaction and fixed relative w.f. $\chi(\rho_i)$

relative coord



Zeroth approximation:

$$\Psi_N\{\boldsymbol{R}_i,\boldsymbol{\rho}_i\} = n\prod_{i=1}^{N/2}\varphi_o(\boldsymbol{R}_i)\zeta(\boldsymbol{\rho}_i) \longleftarrow$$

can usually forget

simple Bose condensation (BEC)

Slightly better approximation:

still treat as structureless bosons, (*i.e*. still ignore $\zeta(\boldsymbol{\rho}_i)$) but allow for interactions:

$$\Psi_N\{\boldsymbol{R}_i\} \neq \prod_i \varphi_o(R_i)$$

Generalized concept of BEC (Yang):

d.f. $\rho_1(\boldsymbol{R},\boldsymbol{R}')$

$$\equiv N\int \Psi_N^*\left(R_1 = R_1, R_2 \ldots R_{N/_2}\right)\Psi_N\left(R_1 = R', R_2 \ldots R_{N/2}\right)$$

$$d\boldsymbol{R}_2 \ldots d\boldsymbol{R}_{N/2}\left(\equiv \left\langle \hat{\psi}^\dagger(\boldsymbol{R})\hat{\psi}(\boldsymbol{R}')\right\rangle_o\right)$$

single-particle density matrix

$$\rho_1(\boldsymbol{R},\boldsymbol{R}') = \sum_i n_i\,\xi_i^*(\boldsymbol{R})\xi_i(\boldsymbol{R}') \qquad \left(\sum_i n_i = N\right)$$

If one and only one of n_i is $O(N)$, not $o(1)$, define for this value of i

$n_i \equiv N_o \equiv$ condensate no. (< N in general)

$\xi_i(R) \equiv \Psi(\boldsymbol{R}) \equiv$ condensate w.f. $\left(O.P. \equiv \sqrt{N_o}\Psi_o(R)\right)$

2. Now weaken interaction:

α_s increases, eventually becomes $\sim r_{int}$.
now can no longer treat as $N/2$ structureless bosons! (can no longer ignore $\zeta(\boldsymbol{\rho}_i)$, nor more importantly underlying Fermi statistics) $\Longrightarrow$ must formulate in terms of fermion coordinates $\boldsymbol{r}_i$ $(i = 1,2 \dots N)$ and spins σ_i, *i.e.*

$$\Psi_{\mathrm{N}} = \Psi_{\mathrm{N}}\,(\boldsymbol{r}_1\,\boldsymbol{r}_2\ \dots\ \boldsymbol{r}_N : \ \sigma_1\,\sigma_2\ \dots\ \sigma_N)$$

But no reason to think “BEC “ goes away!

Generalized definition of (“pseudo -”) BEC (Yang):

define $\rho_2\left(\eta_1\,\Sigma_1, \eta_2\,\Sigma_2 : \eta'_1\,\Sigma'_1, \eta'_2\,\Sigma'_2\right)$

$$\equiv N(N-1)\sum_{\{\sigma_3\dots\sigma_N\}}\int dr_3 \dots dr_N,$$

$$\Psi_N^*(r_1 = \eta_1, \sigma_1 = \Sigma_1, r_2 = \eta_2, \sigma_2 = \Sigma_2 : r_3\sigma_3 \dots r_N\sigma_N) \times$$

$$\Psi_N(r_1 = \eta'_1, \sigma_1 = \Sigma'_1, r_2 = \eta'_2, \sigma_2 = \Sigma'_2 : r_3\sigma_3 \dots r_N\sigma_N)$$

(~ “best description of behavior of 2 particles averaged over that of N-2 others”)

From now on: $\eta_1 \rightarrow r_1, \Sigma_1 \rightarrow \sigma_1$, etc., so

$$\rho_2 \equiv \rho_2(r_1\sigma_1, r_2\sigma_2, : r'_1\sigma'_1, r1_2\sigma'_2)$$

$$(\equiv \langle \hat{\psi}^{\dagger}_{\sigma_1}(r_1)\hat{\psi}^{\dagger}_{\sigma_2}(r_2)\psi_{\sigma'_2}(r'_2\ )\psi_{\sigma'_1}(r'_1)\rangle_o)$$

Since $\rho_2$is Hermitian, can expand:

$$\rho_2(r_1\sigma_1, r_2\sigma_2\colon\ r'_1\sigma'_1, r'_2\sigma'_2) = \sum_i n_i\chi_i^*\,(r_1\sigma_1, r_2\sigma_2)\chi_i(r'_1\sigma'_1, r'_2\sigma'_2)$$

$$\left(\sum_i n_i = N(N-1)\right)$$

Max. of any single eigenvalue n_i is $O(N)$ (not $O\big(N(N-1)\big)$ (Yang)
If one and only one n_i is $O(N)$, rest $O(1)$, define for that value of i

$$n_i \equiv \mathrm{N}_o \equiv \text{condensate number}$$

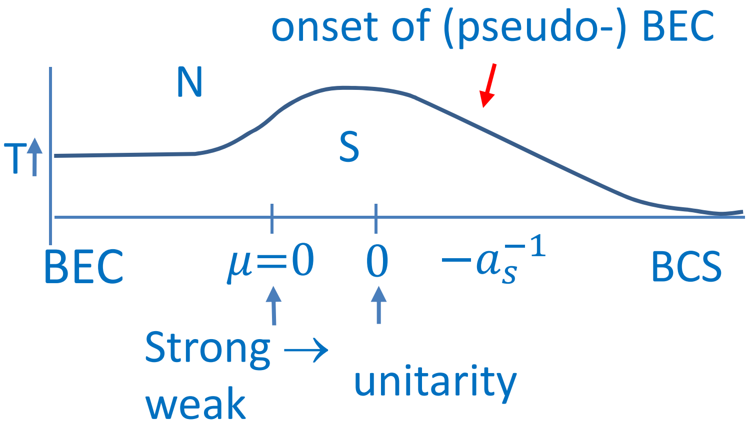
$$\chi_i(r_1\sigma_1, r_2\sigma_2 :) \equiv \chi_o(r_1\sigma_1, r_2\sigma_2) \equiv \text{condensate wave function}$$

(One possible) definition of OP:

$$\mathrm{F}(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) \equiv \sqrt{\mathrm{N}_o}\,\chi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) \leftarrow \text{2 particle quantity!}$$

[Note: at this point, still have $a_s > 0$!]

3. Decrease attraction (increase a_s) further: two qualitatively significant points:
   1. $\mu = 0$ ("strong pairing $\rightarrow$ weak pairing")
   2. $a_s^{-1} = 0$ ("unitarity")



In s-wave case, no qualitative change at either 1 or 2.

Finally, let $|a_s| \rightarrow 0$ $(a_s < 0)$ $\Longrightarrow$ Fermi system with very weak attraction. (BCS problem)

In this limit, N-particle GS believed to be special case of generalized pairing ansatz

all same!

$$\Psi_N^{(pair)} = \mathfrak{N}\mathcal{A}\,\varphi(\boldsymbol{r}_1\sigma_1,\boldsymbol{r}_2\sigma_2)\varphi(\boldsymbol{r}_3\sigma_3,\boldsymbol{r}_4\sigma_4).....\varphi(\boldsymbol{r}_{N-1}\sigma_{N-1},\boldsymbol{r}_N\sigma_N)$$

normalizer antisymmetrizer

formally identical to BEC of molecule!

Note: can still define the "condensate wave function" $\chi(\boldsymbol{r}_1\sigma_1,\boldsymbol{r}_2\sigma_2)$, but (unlike at the BEC end) it is not equal to $\varphi(\boldsymbol{r}_1\sigma_1,\boldsymbol{r}_2\sigma_2)$. (see below)

For illustration, specialize to case (BCS problem)

1. COM at rest $\Rightarrow \varphi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = \varphi(\boldsymbol{r}_1 - \boldsymbol{r}_2, \sigma_1\ \sigma_2)$

2. Spin singlet $\Rightarrow \varphi(\boldsymbol{r}_{-1} - \boldsymbol{r}_{-2}, \sigma_1\ \sigma_2) =$
$$\varphi(\boldsymbol{r}_1 - \boldsymbol{r}_2)\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$$

3. Isotropic $\Rightarrow \varphi(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \varphi(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$

Then straightforward to show* that in 2nd – quantized notation

$$\Psi_N = \mathfrak{N}\left(\sum_{\boldsymbol{k}} c_{\boldsymbol{k}} a^{+}_{\boldsymbol{k}\uparrow} a^{+}_{-\boldsymbol{k}\downarrow}\right)^{N/2} |vac>\ ,$$

$$c_k = \mathrm{F.T.\,of}\ \varphi(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) = f(|\boldsymbol{k}|)$$

Normalization: $$\mathfrak{N} = \frac{1}{N!}\prod_k (1 + |c_k|^2)^{-1/2}$$

with constraint $$\sum_k \left(\frac{|c_k|^2}{1 + |c_k|^2}\right) = N/2$$

*see *e.g.* AJL, Quantum Liquids section 5.4

Two vital quantities:

Recall

$$\Psi_N\{c_k\} \equiv \quad (N!)^{-1}\prod_k (1+|c_k|^2)^{-1/2}\left(\sum_k c_k a^+_{k\uparrow}a^+_{-k\downarrow}\right)^{N/2} \quad |vac\rangle$$

Pick out a particular pair of states $(\boldsymbol{q}\uparrow, -\boldsymbol{q}\downarrow)$ and define

$$\sum_k{}' \equiv \sum_{k\neq q} \quad , \quad \prod_k{}' \equiv \prod_{k\neq q} \quad ,$$

$$\Psi'_N \equiv (N!)^{-1}\prod_k{}'(1+|c_k|^2)^{-1/2}\left(\sum_k{}' c_k a^+_{k\uparrow}a^+_{-k\downarrow}\right)^{N/2} \quad |vac\rangle$$

Then we have $\quad \Psi_N = \dfrac{1}{\sqrt{1+|c_2|^2}}\left(\Psi'_N + c_q a^+_{q\uparrow}a^+_{-q\downarrow}\Psi'_{N-1}\right)$

in words:

amplitude for $\left\{\begin{array}{c} \text{pair in } (q\uparrow, -q\downarrow) \\ \text{no pair in} (q\uparrow, -q\downarrow) \end{array}\right. = \begin{array}{c} c_q/\sqrt{1+\left|c_q\right|^2} \\ 1/\sqrt{1+\left|c_q\right|^2} \end{array}$

Hence

(a) $\langle n_{q\uparrow}\rangle = \langle n_{-q\downarrow}\rangle = \left|c_q\right|^2/\left(1+\left|c_q\right|^2\right)$

(b) $\langle a^+_{q\uparrow}a^+_{-q\downarrow}a_{-q'\downarrow}a_{q'\uparrow}\rangle =$

$$c_q^* c_{q'}/\left(1+\left|c_q\right|^2\right)\left(1+\left|c_{q'}\right|^2\right) \equiv F_q^* F_{q'} \quad ,$$

where $\quad F_q \equiv \dfrac{c_q}{1+\left|c_q\right|^2} \quad \equiv$ “anomalous amplitude”

$$= \langle \Psi_{N-1}\left|a_{-q\downarrow}a_{q\uparrow}\right|\Psi_N\rangle \leq \frac{1}{2}$$

All the above is general, for any choice of c_q’s.

Notes:

1. Normal GS is special choice, with

$$c_k = \Theta(k_F - k)\left(\left(\sum_{k<k_F} a^+_{k\uparrow}a^+_{-k\downarrow}\right)^{N/2} \equiv \prod_{k<k_F} a^+_{k\uparrow}a^+_{-k\downarrow}\right)$$

2. Multiplication of all c_k's by <u>the same</u> phase factor $e^{i\varphi}$ is equivalent to multiplying complete MBWF by $\exp iN\varphi/_2 \Rightarrow$ no physical significance

3. The substitution $c_q \rightarrow -c_q^{*-1}$ produces a paired state orthogonal to $\Psi_N\{c_k\}$, with

$$n_q \rightarrow 1 - n_q, F_q \rightarrow -F_q$$

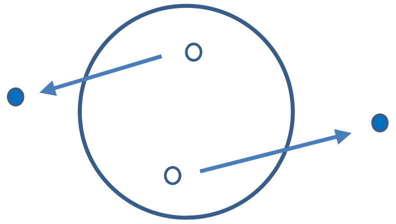
4. An alternative representation of $\Psi_N\{c_k\}$: start from

$$|\mathrm{FS}\rangle \equiv \prod_{k<k_F} (a^+_{k\uparrow}a^+_{k\downarrow})|vac\rangle$$

$$\Psi_N =$$

$$\mathfrak{N}\int_o^{2n} d\varphi \exp\left(e^{i\varphi}\sum_{k>k_F} c_k a^+_{k\uparrow}a^+_{-k\downarrow\Lambda} + e^{i\varphi|}\sum_{k>k_F} d_k a_{-k\downarrow}a_{k\uparrow}\right)|FS\rangle$$

For s-wave case (only!) this is just an alternative (equivalent) way of writing Ψ_N. But…



Relation to Yang's ideas:

At first sight tempting to identify the "single macroscopic eigenvalue" of 2-particle d.m. $\hat{\rho}_2$ as $N/2$ and the corresponding eigenfunction as $\varphi_o(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$. This is right in the BEC limit, but gets progressively worse as we cross over to the BCS limit because of effects of Pauli principle (need to antisymmetrize Ψ_N). Rather, consider

$$\rho_2(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2 \colon \boldsymbol{r}'_1\sigma'_1, \boldsymbol{r}'_2\sigma'_2) \equiv$$

$$\langle \psi^\dagger_{\sigma_1}(r_1)\psi^\dagger_{\sigma_2}(r_2)\psi_{\sigma'_2}(r'_2)\psi_{\sigma'_1}(r_1)\rangle$$

$$= \sum_i n_i \chi_i^*(r_1\sigma_1, r_2\sigma_2)\chi_i(r'_1\sigma'_1 \colon r'_2\sigma'_2)$$

Most intuitive to take F.T.'s and rearrange:

$$\text{F.T.} = \langle a_k^+ a_l^+ a_m a_n \rangle_{N,0} = \langle a_m a_n a_k^+ a_l^+ \rangle_{N,0} + o(N^{-1})$$

Quite generally,

$$\langle a_m a_n a_k^+ a_l^+ \rangle_{N,0} =$$

$$\sum_i \langle N,0| a_m a_n |N+2, i\rangle | \langle N+2, i\ | a_k^+ a_l^+ |N, 0\rangle$$

(i <u>any</u> complete orthonormal set of $N+2$ –particle wave functions).

So question is:

Can we find any $N+2$ −particle state i and any combination $\widehat{\Omega}^{\dagger} \equiv \sum_{kl} c_{kl} a_k^+ a_l^+$ s.t. $\langle N+2, i\,|\widehat{\Omega}|N,0\rangle = 0(N^{1/2})$?

For "normal" state $|N,0\rangle$ (e.g. Fermi sea) this is not possible. But for $|N,0\rangle$ a Cooper-paired state we can choose

$$c_{kl} = \delta_{k,-\ell} \quad i.e. \quad \Omega^{\dagger} = \sum_k a_{k\uparrow}^+ a_{-k\downarrow}^+ \quad ,$$

$i = 0$ (*i.e.* $N+2$ −particle GS)

and then since

$$\langle N+2, 0|a_k^+ a_{-k}^+|N, 0\,\rangle \equiv F_k$$

$$\langle N+2, 0|\Omega^{\dagger}|N, 0\,\rangle \equiv \sum_k F_k$$

Thus the "condensate wave function" $\chi_o(\boldsymbol{r}_1\sigma_1\boldsymbol{r}_2\sigma_2)$ is just the F.T. $(w.r.t.\boldsymbol{r}_1 - \boldsymbol{r}_2)$ of F_k, and the corresponding eigenvalue N_o is $\sum_k |F_k|^2$. In the BCS limit when $F_k = \Delta_k/2E_k$, this quantity is $O\big(\Delta \cdot N(O)\big) \sim N(\Delta/E_F)$, *i.e.*

in BCS limit, condensate fraction $\sim\Delta/E_F$

For the purposes of evaluating pairing contribution to any 2–particle quantity (*e.g.* V), F.T. of F_k, $F(r)$ plays exactly role of 2–particle wave function

$$e.g.\ \ \langle V\rangle = \int V(r)|\underset{\uparrow}{\psi}(r)|^2 dr \Rightarrow \langle V\rangle = \int V(r)|\underset{\uparrow}{F}(r)|^2 dr$$

2 – particle wave function

pair wave function

Which choice of $\{c_k\}$ makes $\Psi_N\{c_k\}$ the groundstate of the N—particle system?

Must minimize $\langle H\rangle \equiv \underset{\substack{\nearrow \\ \text{kinetic en.}}}{\langle T\rangle} + \underset{\substack{\nwarrow \\ \text{potential en.}}}{\langle V\rangle}$ (note since N fixed, no $-\mu N$)

$$\langle T\rangle \equiv \sum_{k\sigma} \underset{\substack{\uparrow \\ \frac{\hbar^2 k^2}{2m}}}{\xi_k} \langle n_{k\sigma}\rangle = 2\sum_k |c_k|^2/(1+|c_k|^2)$$

What about $\langle V\rangle \equiv \sum_{kq\sigma} V_q \left\langle a^+_{k+q/2,\sigma} a_{k-q/2\sigma} a^+_{k'-q/2,\sigma'} a_{k'+q/2,\sigma'}\right\rangle$?

In any completely paired state $\Psi_N\{c_k\}$, only 3 types of nonzero term:

(a) Hartree $(q=0)$: $\langle V\rangle_H = N^2 V_o \neq f\{c_k\} \Rightarrow$ can neglect in minimization.

(b) Fock: $(k'=k-q)\langle V\rangle_F = -\sum_{kq} V_q \langle n_{k+q/2,\sigma} n_{k-q/2,\sigma}\rangle$ in principle affects BCS gap equation, but under most conditions changes little in $N \to S$ transaction, so usually neglected.

(c) Pairing (BCS) $(k=-k')$:

$$\langle V\rangle_{BCS} = \sum_{kk\prime} V_{k-k\prime}\langle a^+_{k\uparrow} a^+_{-k\downarrow} a_{-k\prime\downarrow} a_{k'\uparrow}\rangle = \sum_{kk\prime} V_{k-k\prime} F^*_k F_{k\prime}$$

Thus, minimize

$$\langle H\rangle' = \langle T\rangle + \langle V\rangle_{BCS} = 2\sum_{k} \xi_k \langle n_k\rangle + \sum_{kk'} V_{k-k'} F_k^* F_{k'}$$

with

$$\langle n_k\rangle = \frac{|c_k|^2}{1+|c_k|^2}, \qquad F_k = \frac{c_k}{1+|c_k|^2} \quad ($$

Convenient to note:

$$\left(\sqrt{(1-4|F_k|^2)} - 1\right)^2 = 2(\langle n_k\rangle - \langle n_k\rangle_N)^2$$

$$\langle n_k\rangle_N = \theta(k-k_F)$$

and to subtract a constant term $-E_F N = -\mu N$ from $\langle H\rangle'$, so, $\langle T\rangle_{eff} = 2\sum_k \epsilon_k \langle n_k\rangle$, $\epsilon_k \equiv \xi_k - \mu$. Then minimization yields* a Schrödinger-like equation for F_k

$$\frac{2|E_k|F_k}{\sqrt{1-4|F_k|^2}} + \sum_{k'} V_{kk'} F_{k'} = 0$$

This is just BCS gap equation in disguise! (put $E_k \equiv |E_k|/\sqrt{1-4|F_k|^2}$, $\langle F_k\rangle \equiv \Delta_k/2E_k$)

Note: NO USE OF “SPONTANEOUSLY BROKEN U(1) SYMMETRY”!

* See e.g. *QL* section 5.4

Relation of “particle-conserving” (PC) approach to BCS one:

PC: $$\Psi_N = \mathcal{N}\left(\sum\nolimits_k c_k a_{k\uparrow}^{+} a_{-k\downarrow}^{+}\right)^{N/2} |vac\rangle$$

BCS:

$$\Psi_N \to \mathcal{N}\exp\left(\sum\nolimits_k c_k a_{k\uparrow}^{+} a_{-k\downarrow}^{+}\right)|vac\rangle \equiv \mathcal{N}\prod\nolimits_k (\exp c_k\, a_{k\uparrow}^{+} a_{-k\downarrow}^{+})|vac\rangle$$

$$\Rightarrow \text{(Pauli principle)}$$

$$n\prod_k (1 + c_k a_{k\uparrow}^{+} a_{-k\downarrow}^{+})|vac\rangle \Rightarrow \prod\nolimits_k \mathcal{N}_k\,(1 + c_k a_{k\uparrow}^{+} a_{-k\downarrow}^{+})|vac\rangle$$

with $\mathcal{N}_k = (1 + |c_k|^2)^{-1/2}$

If we write $c_k \equiv \mathrm{v}_k/u_k$ with $|u_k|^2 + |\mathrm{v}_k|^2 = 1$, this becomes

$$\Psi_{\mathrm{BCS}} = \prod_k (u_k + \mathrm{v}_k a_{k\uparrow}^{+} a_{-k\downarrow}^{+})|vac\rangle \equiv \prod_k (u_k|00\rangle_k + \mathrm{v}_k|11\rangle_k)$$

BCS form. $\equiv \Psi_{\mathrm{BCS}}\,\{u_k, \mathrm{v}_k\}$

From $\Psi_{BCS}\{u_k, v_k\}$ we can recover Ψ_N by “Anderson trick”:

$$\mathrm{v}_k \to \mathrm{v}_k \exp i\varphi$$

$$\Psi_N = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, \Psi_{\mathrm{BCS}}(\varphi) \exp iN\varphi/2$$

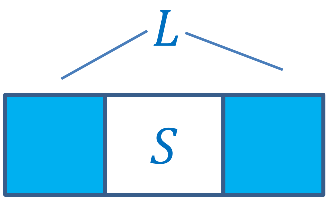
BCS maneuver is equivalent to

$\Psi_{\mathrm{N}} \rightarrow \sum_{N} c_N \Psi_N$, *i.e.* "spontaneous breaking of U(1) gauge symmetry"

Is this ever valid as a description of physical state? NO!!

Superselection rule for particle number prohibits it!

Digression: What if system has leads?



Then indeed N_s is not conserved, but $N_s + N_i$ is, $= N_{tot}$ (say), so

$$\Psi = \Psi(N_s, N_L)$$
$$= \sum_{N_S} C_{N_S} \Psi_S\,(N_s\,)\Psi_L(N_{tot}\, - N_s)$$

so reduced density matrix of S (obtained by tracing over N_L) still diagonal in N_s −representation:

$$\rho_{N_s,N_s'} \sim f(N_s)\delta_{N_s,N_s'}$$

⇒ "spontaneous breaking of U(1) symmetry" IS NOT PHYSICAL!

Final note: BCS ansatz for CS is inconsistent!

Take a neutral Fermi system and consider the quantity

$$S(q) \equiv \langle \hat{\rho}_q \hat{\rho}_{-q} \rangle \qquad (q \neq 0)$$

$$\hat{\rho}_q \equiv \sum_{k\sigma} a^+_{k+q/2,\sigma} a_{k-q/2,\sigma}$$

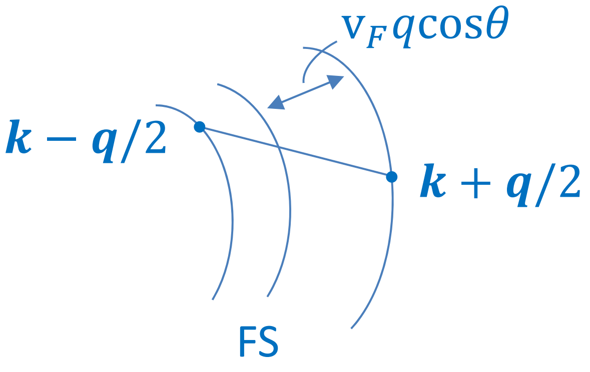
Assuming compressibility of system (N/mc^2) is not infinite, f −sum rule + compressibility sum rule (KK) + Cauchy-Schwarz ⇒

$$S(q) \leq Nq/mc. \qquad (q \to o)$$

For a free Fermi gas, $S(q)$ has only the "Fock" contribution

$$S_F(q) = \sum_{kr} \left\langle n_{k+q/2,\sigma} \left(1 - n_{k-q/e^{2}\prime\sigma}\right)\right\rangle = \frac{3N}{2mv_F} q$$

and since $c = \mathrm{v}_F/\sqrt{3}$, inequality is satisfied.



But for BCS groundstate there is also a pairing term:

$$S_{\text{pair}}(q) = \sum_k \frac{\Delta^*_{k+q/2}}{2E_{k+q/2}} \frac{\Delta_{k-q/2}}{2E_{k-q/2}} \overset{\Rightarrow}{(q \rightarrow O)} \sum_k \frac{|\Delta_k|^2}{4E_k^2} \sim N(\Delta/E_F)$$

So for $q \lesssim mc(\Delta/E_F)(\sim\xi^{-1})$, inequality is violated!

Solution: must build into GSWF zero-point density fluctuations! (i.e. zero-point AB modes)

↑ Anderson-Bogoliubov

Intuitively: BCS GS⇒ φ =constant. But this then implies huge fluctuations in condensate no. density ⇒ huge repulsion energies. ZP AB modes "smooth out" density!

[For a charged system (metal), problem is "hidden" because it already occurs in the N phase and is taken into account by involving ZP plasmons.]