Cooper Pairing in "Exotic" Fermi Superfluids: An Alternative Approach

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Lecture 2

1. Anisotropic Cooper pairing (textbook version)

Can still use fundamental ansatz (*l*. 1)

same φ!

$$\Psi_N(\boldsymbol{r}_1\sigma_1 \ldots \boldsymbol{r}_N\sigma_N) =$$

$$n \cdot \mathcal{A} \cdot \varphi(\boldsymbol{r}_1\sigma_1\boldsymbol{r}_2\sigma_2)\varphi(\boldsymbol{r}_3\sigma_3\boldsymbol{r}_4\sigma_4)\ldots\, \varphi(\boldsymbol{r}_{N-1}\sigma_{N-1}\boldsymbol{r}_N\sigma_N)$$

normalization (n), antisymmetrizer ($\mathcal{A}$)

and still require

$$\varphi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = \varphi(\boldsymbol{r}_1 - \boldsymbol{r}_2 : \sigma_1\, \sigma_2) \quad \leftarrow \textit{i.e.}\ \text{COM at rest}$$

but now allow φ to be nontrivial functions of σ's and $\hat{\boldsymbol{r}}_{12}$.

However, because of $\mathcal{A}$ (Pauli principle) $S = 0 \Leftrightarrow f(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ even, $S = 1 \Leftrightarrow f(\boldsymbol{r}_1 - \boldsymbol{r}_2)$ odd.

Even-parity case is simple generalization of BCS:

$$\Psi_N = \mathcal{N}\left(\sum_k c_k a^+_{k\uparrow} a^+_{-k\downarrow}\right)^{N/2} |vac\rangle \qquad c_{\boldsymbol{k}} = c_{-\boldsymbol{k}} = f\left(|\boldsymbol{k}|, \hat{\boldsymbol{k}}\right)$$

$\Rightarrow$ generalized gap equation

$$\Delta_k = \sum_{k\prime} V_{kk\prime}\, \Delta_{k\prime}/2E_{k\prime} \qquad E_{k\prime} \equiv \left(E_k^2 + |\Delta_k|^2\right)$$

in general (in 3D) gap has nodes at 2 or more points on F.S.

Odd-parity case: must now pair spins to form $S = 1$. Description of general case (*e.g.* $^3He - B$) complicated, but simplifies for ESP ($^3He - A, Sr_2RuO_4$). In this case proper description is

equal spin pairing
($\uparrow\uparrow$ and $\downarrow\downarrow$, no $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$)

$$\varphi(\boldsymbol{r}_1 - \boldsymbol{r}_2 : \sigma_1\, \sigma_2) = f_{\uparrow\uparrow}(\boldsymbol{r}_1 - \boldsymbol{r}_2)|\uparrow\uparrow\rangle + e^{i\varphi} f_{\downarrow\downarrow}(\boldsymbol{r}_1 - \boldsymbol{r}_2)|\downarrow\downarrow\rangle$$

$$\Rightarrow \Psi_N = \mathcal{N}\left(\sum_k c_{k\uparrow} a^+_{k\uparrow} a^+_{-k} + e^{i\varphi} c_{k\downarrow} a^+_{k\downarrow} a^+_{-k\downarrow}\right)^{N/2} |vac\rangle \qquad (\uparrow !)$$

but in most TQC contexts usually adequate to replace by

$$\Psi_N = \Psi_{\frac{N}{2}\uparrow} \Psi_{\frac{N}{2}\downarrow} \quad , \quad \Psi_{\frac{N}{2}\uparrow} = \mathcal{N}_\uparrow \left(\sum_k c_k a^+_{k\uparrow} a^+_{-k\uparrow}\right)^{N/4} |vac\rangle \quad \text{(etc.)}$$

From now on, concentrate on single spin population (*e.g.* $\uparrow$) so omit $\uparrow$'s and let $N/2 \to N$ (*i.e.* N is number of "relevant" particles), $\Psi_{N/2,\uparrow} \to \Psi_N$

Thus, at first sight,

$$\Psi_N = \mathcal{N}\left(\sum_k c_k a^+_{\boldsymbol{k}} a^+_{-\boldsymbol{k}}\right)^{N/2} |vac\rangle \qquad c_{\boldsymbol{k}} = -c_{-\boldsymbol{k}} \qquad \text{(Pauli)}$$

$\uparrow$: State $\boldsymbol{k} = 0$ has no partner! (contrast even-parity case, $(O \uparrow O \downarrow)$), hence, strictly correct odd-parity GSWF for an odd number $(2N + 1)$ of particles is

$$\Psi_N = \mathcal{N}\left(\sum_{\boldsymbol{k}\neq 0} c_k a^+_k a^+_{-k}\right)^{N/2} a^+_o |vac\rangle$$

(We will usually implicitly subtract off this odd particle in the accounting)

Most interesting case (especially in 2D) is when c_k is (nontrivially) complex. Important example $\left({}^3He - A, Sr_2RuO_4(?) \right)$ is $p + ip$ state:

$$c_k = f(|\boldsymbol{k}|) \exp i\varphi_k \quad \leftarrow \angle \text{ of } \boldsymbol{k} \text{ on Fermi surface}$$

In this case, pair wave function F_k is of form

$$F_k = \frac{c_k}{1 + |c_k|^2} = \frac{\Delta_k}{2E_k} \qquad \Delta_k = |\Delta_k| \exp i\varphi_k$$

(so in 2D, no nodes). Thus, if we write

$$|\Delta_k|_{|k|=k_F} \equiv \Delta_o k_F \ ,$$

then on and near F.S.,

$$\Delta_k = \Delta_o (k_x + ik_y) \qquad \text{(hence, “}p + ip\text{”)}$$

However, from symmetry of gap equation (with $V_{kk\prime} = V_{\boldsymbol{k}-\boldsymbol{k}\prime}$)

$$\Delta_k = -\sum_{k\prime} V_{\boldsymbol{k}-\boldsymbol{k}\prime} \Delta_{k'} | 2E_{k\prime}$$

$$\left(E_k \equiv \left(\epsilon_k^2 + |\Delta_k|^2\right)^{1/2} = f(|\boldsymbol{k}|)\right.$$

Δ_k must have same <u>symmetry</u> for all $|k|$ (down to $|k| \to 0$). Does it also have same dependence on $|k|$? (*i.e.* is $|\Delta_k| \propto |k|$? If potential $V(r)$ can be Taylor-expended, yes – usually assumed in literature. Thus, usual assumption is

$$\Delta_k = \Delta_o\left(k_x + ik_y\right), \quad \forall \boldsymbol{k} \quad \text{(including } |k| \to 0\text{).}$$

For future simplicity, write for 2D $p + ip$ state

$$\widehat{\Omega}^\dagger \equiv \mathcal{N}' \sum_k |c_k| e^{i\varphi_k} a_k^+ a_{-k}^+$$

so that

$$\Psi_N^{p+ip} = \left(\Omega^\dagger\right)^{N/2} |vac\rangle$$

The 2D $p + ip$ state has several intriguing properties:

1. Since by direct calculation $[\hat{L}_z\,, \hat{\Omega}^\dagger] = \hbar\Omega^\dagger$

we have

$$\hat{L}_z|\Psi_N\rangle = \frac{N\hbar}{2}|\Psi_N\rangle$$

2. Nevertheless, if we take the limit $|c_k| \to \theta(k_F - k)$, we find

$$\Psi_N = \left(\sum_{k<k_F} e^{i\varphi_k} a_k^+ a_{-k}^+\right)^{N/2} |vac\rangle$$

$$= \prod_{k<k_F} \left(e^{i\varphi_k} a_k^+ a_{-k}^+\right) |vac\rangle$$

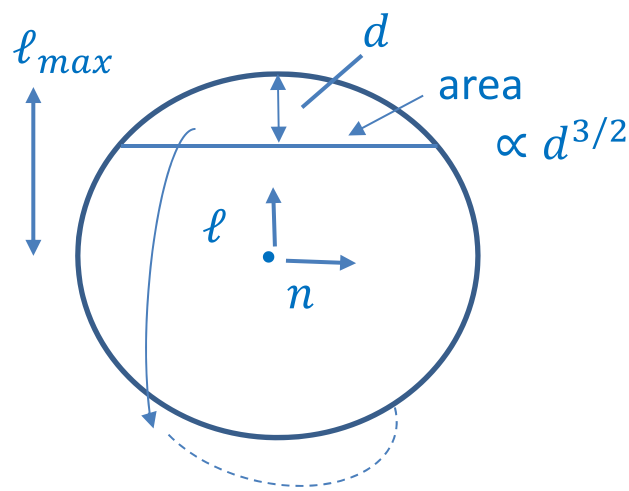
$$\equiv \left(\prod_{k\angle k_F} e^{i\varphi_k}\right) \prod_{k\angle k_F} a_k^+ a_{-k}^+ |vac\rangle \equiv \left(\prod_{k<k_F} e^{i\varphi_{k'2}}\right) |FS\rangle$$

So since an overall phase factor is physically irrelevant, $\hat{L}_z|\Psi_N\rangle = 0$!

Relevant observation: how much energy does it cost to polarize Fermi sea to ang. momentum $N\hbar/2$?

Answer: $0\left(N^{1/3}\right)$!

(Moment of inertia is not extensive)



Possible resolution of apparent inconsistency of 1 and 2 (M. Stone):

In a finite system of dimension R as $\Delta \rightarrow 0$, Cooper-pair radius $\xi \sim \hbar v_F/\Delta$ eventually becomes $> R$. So, for $\xi \ll R \quad L \sim N\hbar/2$, for $\xi \gg R \;\; L \sim 0$?

3. Behavior of "molecular" wave function φ at large $|\boldsymbol{r}_1 - \boldsymbol{r}_2|$:

By inverting the general relation $\mathrm{F}_k = \frac{c_k}{1+|c_k|^2}$ and

substituting the equilibrium value of $\mathrm{F}_k, \Delta_k/2\mathrm{E}_k$,

we obtain for $k < k_\mathrm{F}(\ \epsilon_k < \mu)$ the relation

$$c_k = \frac{\Delta_k/2\mathrm{E}_k}{\left(1 - \frac{|\epsilon_k - \mu|}{\mathrm{E}_k}\right)}$$

which in the limit $\epsilon_k \to 0$ reduces to $\mu\ /\ 4\Delta_k^*$ $(+0(\Delta^2\ /\ \mu^2)$. Thus provided $|\Delta(k)| \propto |k|$ for $|k| \to 0$ as assumed,

$$c_k = \frac{const}{k_x\ -\ ik_y} \qquad as\ k \to 0$$

Thus the F. T., the "molecular wave function" $\varphi(\boldsymbol{r})(\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2)$, behaves at large differences as

$$\varphi(r) \sim z^{-1} \qquad (z \equiv x + iy)$$

and for these distances the many body GSWF has the "Pfaffian" form

$$\Psi_\mathrm{N} \sim \mathrm{P}f \left\{\frac{1}{z_i - z_j}\right\}$$

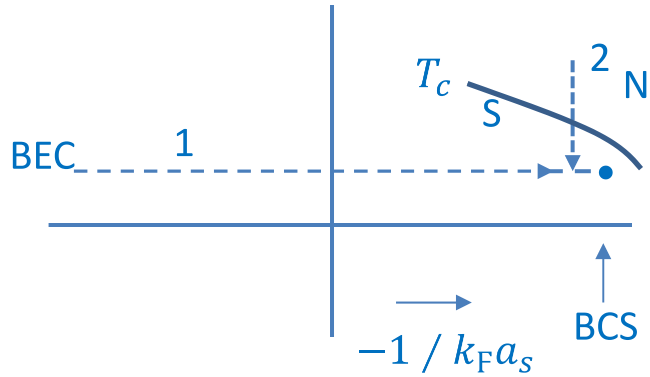
←Moore – Read form for $\nu = 5/2$ QHE

2. Anisotropic Cooper pairing: alternative approach

The ground state of a $p + ip$ Fermi superfluid looks "natural" when reached by path 1, much less so when matched by 2, since

(a) $\Delta L\ at\ T_c \sim \mathrm{N}$

(b) Cooper instability affects only states near Fermi surface, not far down in Fermi sea. So consider the ansatz

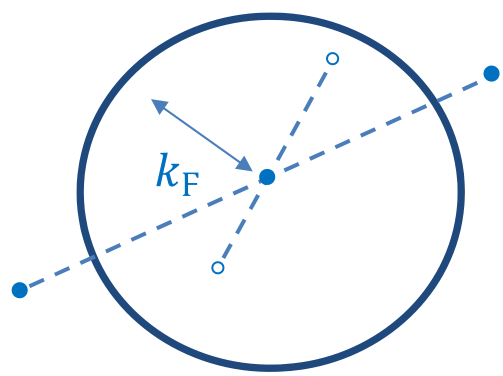


$$\Psi_{\mathrm{N}}^{alt} \sim n\left(\sum_{k>k_{\mathrm{F}}} c_k a_k^+ a_{-k}^+\right)^{\mathrm{M}} \left(\sum_{k<k_{\mathrm{F}}} d_k a_{-k} a_k\right)^{\mathrm{M}} |FS\rangle$$

$$(\mathrm{M} \ll \mathrm{N})$$

where d_k is related to the c_k of the textbook approach (call it $c_k^{(0)}$) by

$$d_k = \left(c_k^{(0)}\right)^{-1}$$



This reproduces the "standard" value of $\langle n_k \rangle$, since the number of holes in state $k < k_{\mathrm{F}}$ is $\frac{|d_k|^2}{1+|d_k|^2}$:

$$\langle n_k \rangle = 1 - \frac{|d_k|^2}{1 + |d_k|^2} = \frac{\left|c_k^{(0)}\right|^2}{1 + \left|c_k^{(0)}\right|^2}$$

It also reproduces the value of $\langle a_k^+ a_{-k}^+ a_{-k'} a_{k'} \rangle$ for $|\boldsymbol{k}|, |\boldsymbol{k}'|$ both above or both below k_{F}, but not for (*e.g.*) $k > k_{\mathrm{F}}, k' < k_{\mathrm{F}}$. However this can be remedied by a small modification of Ψ_{N}:

put $\Omega_+^\dagger \equiv \sum_{k>k_F} c_k a_k^+ a_{-k}^+,\ \Omega_-^\dagger \equiv \sum_{k<k_F} d_k a_{-k} a_k$ and write

$\Psi_N^{alt} \equiv \sum_M k_M \left(\Omega_+^\dagger \Omega_-^\dagger\right)^M |\mathrm{FS}\rangle, k_M$ slowly varying in phase as $f(M)$

(one possible implementation:

$$\Psi_N^{(alt)} \sim \int d\varphi \left(\exp i\varphi/2 \cdot \Omega_+^\dagger + \exp -i\varphi/2 \cdot \Omega_-^\dagger\right)^{N/2} |FS\rangle \quad)$$

then both $\langle n_k \rangle$ and $\langle a_k^+ a_k^+ a_{-k'} a_{k'} \rangle$ same for all $\boldsymbol{k}, \boldsymbol{k}'$(to order $N^{-1/3}$)

as in textbook approach $\left(F_k(k < k_F) = \frac{d_k^*}{1+|d_k|^2} = \frac{c_k^{(o)}}{1+\left|c_k^{(0)}\right|^2} \right) \Longrightarrow$

energy/particle of 2 states identical in thermodynamic limit.

At first sight Ψ_N^{alt} is just a rewriting of Ψ_N in different notation, as in the s-wave case. However,

$[L_z, \Omega_+^\dagger] = \hbar \Omega_+^\dagger$ but $[L_2, \Omega_-^\dagger] = -\hbar \Omega_-^\dagger$

$\Longrightarrow \hat{L}_z |\Psi_N^{(alt)}> = 0$! $(+0(\Delta/E_F)^2$due to shift of μ in S phase)

So … which is right, Ψ_N or $\Psi_N^{(alt)}$ (or maybe a hybrid wave function)?

A possible answer: both, or neither! The MBGS may be degenerate within terms of relative order $N^{-1/3}$ in thermodynamic limit.

Note: $\Psi_N^{(alt)}$ almost certainly corresponds to a quite different behavior of $\varphi(r)$ in the limit $r \to \infty$. Plausibly (but not proved): the modification to $\varphi(r)$ from its N −state value $\propto r^{-3}$.

3. <u>Fermionic quasiparticles: the textbook approach</u>
   (BCS (isotropic) case)

Recall: in BCS formalism, even-no-parity (PNC) GS is written in form

$$\Psi_{\mathrm{BCS}} = \prod_k (u_k + v_k a^+_{k\uparrow} a^+_{-k\downarrow})|vac>.$$

$$\equiv \prod_k \Phi_k \,, \Phi_k \equiv u_k|00\rangle_k + v_k\,|11\rangle_k$$

$$u_k = u_{-k}\,, v_k = v_{-k}, u_k \text{ real}$$

We make standard Bogoliubov-Valatin transformation:

$$\alpha^+_{k\sigma} = u_k a^+_{k\sigma} - \sigma v^*_k a_{-k,-\sigma}$$

So (*e.g.*)

$$\alpha^+_{k\uparrow} = u_k a^+_{k\uparrow} - v^*_k a_{-k\downarrow}$$

and so

$$a^+_{k\uparrow}|\Phi_k\rangle \equiv (u_k a^+_{k\uparrow} - v^*_k a_{-k\downarrow})(u_k + v_k a^+_{k\uparrow} a^+_{-k\downarrow})|\mathrm{GS}\rangle$$

$$= u_k^2 a^+_{k\uparrow}|vac\rangle + |v_k|^2 a^+_{k\uparrow}|\mathrm{GS}\rangle$$

ACR's!

$$\equiv \alpha^+_{k\uparrow}\,|GS\rangle \equiv |1{,}0\rangle$$

Similarly the operator

$$\alpha^{+}_{-k\downarrow} \equiv u_k a^{+}_{-k\downarrow} + v^{*}_{k} a_{k\uparrow}$$

when applied to the BCS groundstate, generates the state $|01\rangle_k$.

However, from the four operators $a^{+}_{k\uparrow}, a^{*}_{k\uparrow}, a^{+}_{-k\downarrow}, a_{-k\downarrow}$ one can generate two other linearly independent (orthogonal) combinations:

$$\beta_{k\uparrow} \equiv v_k a^{+}_{k\uparrow} + u_k a_{-k\downarrow}$$

$$\beta_{-k\downarrow} \equiv v_k a^{+}_{-k\downarrow} - u_k a_{k\uparrow}$$

Appling these to the BCS groundstate gives, *e.g.*,

$$\beta_{k\uparrow}|\Phi_k\rangle \equiv (v_k a^{+}_{k\uparrow} + u_k a_{-k\downarrow})(u_{k^+} v_k a^{+}_{k\uparrow} a^{+}_{-k\downarrow})|\mathrm{GS}\rangle$$

$$= v_k u_k a^{+}_{k\uparrow}\,|vac\rangle - u_k v_k a^{+}_{k\uparrow}|\mathrm{GS}\rangle \equiv 0$$

and similarly for β_{k2}. So β_{k1} and β_{k2} are pure annihilators. At first sight this looks trivial, since $\beta_{k\uparrow}$ is just the H.C. of $\alpha^{+}_{-k\downarrow}$, *i.e.* $\alpha_{-k\downarrow}$, and we are all used to the fact that the $\alpha's$ annihilate the ground state. But it's worth noting for future reference…

To summarize, in simple BCS theory the simplest MB energy eigenstates can be expressed in the form of a tensor product of occupation states $\Phi_k$referring to the pair of plane-wave states $(\boldsymbol{k}\uparrow, -\boldsymbol{k}\downarrow)$. The (even-numbered-parity) "ground pair" state is

$$\Phi_k^{GP} = u_k|00\rangle + v_k|11\rangle$$

There is a second "completely-paired" (even-numbered parity) state orthogonal to Φ_k^{GP},:

$$\Phi_k^{EP} = -v_k^*|00\rangle + u_k|11\rangle$$

which can in fact be generalized by successive application of $\alpha_{k\uparrow}^+$ and $\alpha_{-k\downarrow}^+$

The odd-number-parity energy eigenstates are

$$|10\rangle \equiv \alpha_{k\uparrow}^+|\Phi_k^{GP}\rangle \quad \text{and} \quad |01\rangle \equiv \alpha_{-k\downarrow}^+|\Phi_k^{GP}\rangle$$

while the operators $\beta_{k\uparrow} \equiv \alpha_{-k\downarrow}$ and $\beta_{-k\downarrow} \equiv \alpha_{k\uparrow}$ annihilate Φ_k^{GP}

All the above analysis generalizes straightforwardly to the anisotropic case, including the $(p + ip)$ state. (irrespective of whether we use the textbook or alternative approach*).

*In the latter, Φ_k is simply $|u_k|e^{-i\varphi_k}|00\rangle + |v_k||11\rangle$.

So far, so good… But what if our Hamiltonian is more general:

$$\hat{H} = \sum_{\sigma\sigma'} \int dr \left(-\frac{\hbar^2}{2m} \delta_{\sigma\sigma'} \psi_\sigma^\dagger(\boldsymbol{r}) \nabla^2 \psi_{\sigma'}(\boldsymbol{r}) + U_{\sigma\sigma'}(\boldsymbol{r}) \psi_\sigma^\dagger(\boldsymbol{r}) \psi_{\sigma'}(\boldsymbol{r}) \right)$$

$$+\frac{1}{2} \sum_{\alpha\beta\gamma\delta} \iint d\boldsymbol{r}\, d\boldsymbol{r}' V_{\alpha\beta\gamma\delta}(\boldsymbol{r}, \boldsymbol{r}') \psi_\alpha^\dagger(\boldsymbol{r}) \psi_\beta^\dagger(\boldsymbol{r}') \psi_\gamma(\boldsymbol{r}') \psi_\delta(\boldsymbol{r}) \quad ?$$

We can still use our general definition of a "completely paired" N(=even) – particle state:

same!

$$\Psi_N = \mathcal{N} \cdot \mathcal{A} \cdot \{\varphi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)\varphi(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\ldots \varphi(\boldsymbol{r}_{N-1}\sigma_{N-1,} \boldsymbol{r}_N\sigma_N)\}$$

$$\equiv \mathcal{N}' \left(\sum_{\sigma\sigma'} \iint d\boldsymbol{r}\, d\boldsymbol{r}' \varphi(\boldsymbol{r}\boldsymbol{\sigma}, \boldsymbol{r}'\sigma') \psi_\sigma^\dagger(\boldsymbol{r}) \psi_{\sigma'}^\dagger(\boldsymbol{r}') \right)^{N/2} |vac\rangle$$

Theorem: can always find complete orthonormal set $n, \bar{n}$ ($i.e.$ $\langle n, \bar{n}' \rangle = 0$, $\langle n | n' \rangle = \delta_{nn'}$, $\langle \bar{n}, \bar{n}' \rangle = \delta_{\bar{n}\bar{n}'}$) such that

$$\varphi(\boldsymbol{r}\boldsymbol{\sigma}, \boldsymbol{r}'\sigma') = \sum_n c_n \chi_n(r\sigma) \chi_{\bar{n}}(r'\sigma')$$

Thus, can write any completely paired state in the form

$$\Psi_N = \mathcal{N}'' \left(\sum_m c_m a_m^+ a_{\bar{m}}^+ \right)^{N/2} |vac\rangle$$

As long as we deal only with the even-number-parity states, we can continue the analogy to BCS:

$$\bar{\Psi}_{\text{BCS}} = \prod_m \Phi_m^{GP} \, , \;\; \Phi_m^{GP} \equiv u_m|00\rangle + v_m|11\rangle \, , |00\rangle \equiv |00\rangle_{m\bar{m}} \text{ , etc.}$$

$$c_m \equiv v_m/u_m \, , \;\; |u_m|^2 + |v_m|^2 = 1$$

and the "excited-pair" state is given by

$$\Phi_m^{EP} = -v_m^*|00\rangle + u_m|11\rangle \qquad (c_m \to -c_m^{*-1})$$

This is an energy eigenstate if Φ_m^{GP} is, since $\langle n_m \rangle \to 1 - \langle n_m \rangle$ and $F_m \to -F_m$.

In the BCS case we found the values of the c_k's (or equivalently of the quantities $\langle n_k \rangle$ and F_k) by minimizing the sum of the single – particle and pairing terms in $\langle \hat{H} \rangle$. Can we do the same here? Since the functions $\chi_n, \chi_{\bar{n}}$ are unknown a priori, our calculation should also find them. Suppose we write

$$U_m(r) \equiv U_m \chi_m(\boldsymbol{r}\sigma) \, , \qquad V_m(\boldsymbol{r}) \equiv V_m \chi_{\bar{m}}(\boldsymbol{r}\sigma)$$

and minimize the s.p. + pairing terms with respect to the functions $U_m(\boldsymbol{r}\sigma)$, $V_m(\boldsymbol{r}\sigma)$, subject to the orthogonality constraints. $\left((U_m, U_{m'}) \sim \delta_{mm'} \text{ , etc.}\right)$. The resulting equations are, formally, exactly the standard BdG equations (see below) but with much stronger orthogonality constraints. So we can obtain an explicit solution this way only in special cases ($e.g.$ $\bar{m} = \text{TR of } m$).

Now let's turn to the odd-number-parity states. By exact analogy with the BCS case, the combinations

$$\beta_m \equiv v_m a_m^+ + u_m a_{\bar{m}}$$

$$\beta_{\bar{m}} \equiv v_m a_{\bar{m}}^+ - u_m a_{\bar{m}}$$

are pure annihilators: $\beta_m|\Psi_N\rangle = \beta_{\bar{m}}|\Psi_N\rangle = 0$ Note that any linear combination of PA's is itself a PA! Moreover the states

$$\alpha_m^+ \equiv u_m a_m^+ - v_m^* a_{\bar{m}} \equiv \left(\bar{\beta}_{\bar{m}}\right)^\dagger$$
$$\alpha_{\bar{m}}^+ \equiv u_m a_{\bar{m}}^+ + v_m^* \alpha_m \equiv \left(\beta_m\right)^\dagger$$

create the states $|1{,}0\rangle_m$ and $|0{,}1\rangle_m$ respectively $|1{,}0\rangle_m \equiv$ occupied, $\bar{m}$ empty, etc.) and $\alpha_m^+ \alpha_{\bar{m}}^+$ generates the "excited pair" state Φ_m^{EP}. However, these two states are not in general energy eigenstates. This is fairly obvious, since when written in the (basis of the χ_m and $\chi_{\bar{m}}$ even the single-particle term $\widehat{H}$ is nondiagonal. So the odd-parity energy eigenstates must be linear combinations of the states $|1{,}0\rangle_m$ and $|0{,}1\rangle_m$:

$$\gamma_i^\dagger = \sum_m \left(\lambda_{im} \alpha_m^+ + \mu_{im} \alpha_{\bar{m}}^+\right)$$

How to find these linear combinations? Standard method is mean-field (BdG) approach: in spirit of BCS, factorize potential term in Hamiltonian:

$$V\psi_\alpha^\dagger(r)\psi_B^\dagger(r')\psi_\gamma(r')\psi_\delta(r) \to \left\langle \psi_\alpha^\dagger(r)\psi_\beta^\dagger(r')\psi_\gamma(r')\psi_\delta(r)\right\rangle$$
$$+\left\langle \psi_\gamma(r')\psi_\delta(r)\right\rangle \psi_\alpha^\dagger(r)\psi_\beta^\dagger(r')\Big\rangle$$

and define

$$\Delta_{\alpha\beta}(r,r') \equiv \sum_{\gamma\delta} V_{\alpha\beta\gamma\delta}(\boldsymbol{r},\boldsymbol{r}')\left\langle \psi_\gamma(r')\psi_\delta(\boldsymbol{r})\right\rangle$$

Then the effective mean-field Hamiltonian is bilinear in ψ and $\psi^\dagger$:

$$\hat{H}_{mf} = \sum_{\alpha\beta} \int dr \left(\psi_\alpha^\dagger(r) \hat{H}_o \psi_\beta\,(r) \right)$$

$$+\frac{1}{2} \sum_{\alpha\beta} \iint drdr' \left\{ \Delta_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}')\psi_\alpha^\dagger(r)\psi_\beta^\dagger(r') + H.C. \right\}$$

where the quantity $\langle \psi_\gamma(r')\psi_\delta(r) \rangle$ occuring in $\Delta_{\alpha\beta}$must eventually be determined self-consistently. We now seek the creation operators of odd-parity energy eigenstates in the form

$$\gamma_i^\dagger = \int dr \left\{ u(r)\psi^\dagger(r) + v(r)\psi(r) \right\}$$

with the normalization constraint

$$\int |u(r)|^2 + |v(r)|^2 dr = 1$$

in words, we create an extra particle with wave function $u(r)$ and an extra hole with wave function$v(r)$. Demanding that

$$[\hat{H}_{mf}, \gamma_i^\dagger] = E_i \gamma_i^\dagger$$

yields the famous Bogoliubov-de Gennes (BdG) equations, which in general, because of the spin degree of freedom, are 4×4 : here I give for simplicity the version for a single spin species (but keep the spatial "nonlocality"):

$$\hat{H}_o u_i(\boldsymbol{r}) + \int \Delta(\boldsymbol{r},\boldsymbol{r}')\, v_i(\boldsymbol{r}') dr' = E_i u_i(\boldsymbol{r})$$

$$\int \Delta^*(\boldsymbol{r},\boldsymbol{r}')\, u_i(\boldsymbol{r}') dr' - \hat{H}_o^* v_i(\boldsymbol{r}) = E_i v_i(\boldsymbol{r})$$

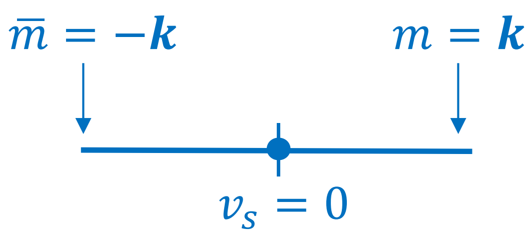
or schematically

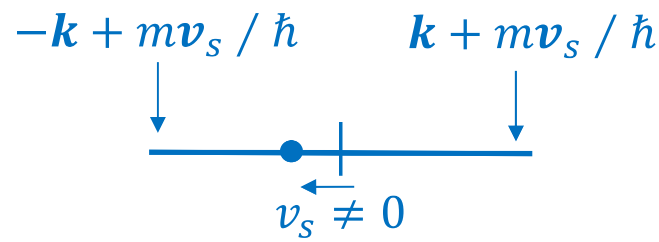
$$\begin{pmatrix} \hat{H}_o & \Delta \\ \Delta & -\hat{H}_o^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = E \begin{pmatrix} u \\ v \end{pmatrix}$$

While the u's do not in general by themselves form an orthonormal set, the spinors (u,v) do, *i.e.* $(u_i,u_j) + (v_i,v_j) = \delta_{ij}$ (generated by equations themselves). Hence the set of $\gamma_i^\dagger$ form a complete set of anticommuting Fermi operators, with $\gamma_i|\Psi_{\mathrm{BCS}}\rangle = 0$.

It is often pointed out in the literature that if $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ solves the BdG equations with energy eigenvalue E_i, then $\begin{pmatrix} -v_i^* \\ u_i^* \end{pmatrix}$ solves them with eigenvalue $-E_i$, and so one says that this combination creates a "negative-energy state". But this is illusory: the corresponding operator (call it $\tilde{\gamma}_i^\dagger$) is actually a combination of pure annihilators and hence itself simply a pure annihilator (in fact, with a π rotation of v relation to u, it is just γ_i).

Problems with the MF-BdG approach:
(a) Galilean invariance: (simple BCS problem)





Rest frame of condensate

$$\alpha_m^+ = u_k a_{k\uparrow}^+ - v_k^* a_{-k\downarrow}$$

$$\mathrm{P} = \hbar\big(|u_k|^2 \boldsymbol{k} - |v_k|^2(-\boldsymbol{k})\big)$$

$$= \hbar \boldsymbol{k} \ \surd$$

$(\equiv \boldsymbol{P} - \boldsymbol{P}_0)$

Total momentum of groundstate $(= 0)$

Frame moving with velocity v_s w.r.t. condensate:

$$(\alpha_m^+)' = u_k a_{k+mv_s/\hbar}^+ - v_k^* a_{-k+mv_s/\hbar}$$

$$(\boldsymbol{P} - \boldsymbol{P}_0)' = \hbar(|u_k|^2 k + mv_s/\hbar)$$
$$-|v_k|^2(-k + mv_s/\hbar))$$

$$= \hbar k + (|u_k|^2 - |v_k|^2) m v_s$$

But: Galilean invariance requires simply

$$(P - P_0)' = (P - P_0) + m\mathrm{v}_s \ !!$$

note discrepancy depends on u_k, v_k, hence cannot be fixed simply by involving "spontaneously broken $U(1)$ symmetry".

Solution: when creating "hole" component of odd-parity state,

MUST ADD A COOPER PAIR!

thus, correct "Bog QP" creation operator is

$$\alpha_m^+ = u_m\, a_m^+ - v_m^* a_{\bar{m}}\, \hat{C}^\dagger \quad \left(\hat{C}^\dagger \equiv \mathcal{N} \sum_m c_m\, a_m^+\, a_{\bar{m}}^+\right)$$

For condensate at rest, changes nothing (still have $(\boldsymbol{P} - \boldsymbol{P}_o) = \hbar\mathbf{k}$).
But for condensate moving, adds an extra 2 $|v_k|^2\, mv_s$, so that

$$(\boldsymbol{P} - \boldsymbol{P}_o)' = \hbar \begin{Bmatrix} |u_k|^2\, (\boldsymbol{k} + m\boldsymbol{v}_s/\hbar) - |v_k|^2\, (-\boldsymbol{k} + m\boldsymbol{v}_{-s}/\hbar) + \\ 2|v_k|^2\, m\boldsymbol{v}_{s/\hbar} \end{Bmatrix}$$

$$= \hbar\boldsymbol{k} + \mathrm{m}\boldsymbol{v}_s = (P - P_o) + \mathrm{m}\boldsymbol{v}_s$$ as required.

(6) NMR of Majorana fermions (M.A. Silaev, PRB **84** 144508 (2011)): consistent calculation based on MF Hamiltonian $\rightarrow$ spectral weight in longitudinal resonance absorption above Larmor frequency. (independent of dipole coupling constant g_D)

Problem: violates sum rule! (for g_D= 0, $\int \omega\chi''(\omega)d\omega = 0$, and for nonzero $g_D, \propto g_D$)

Solution: need to consider response also of added Cooper pair (E. Taylor et al., arXiv:1412.7153)

Moral: In any situation where Cooper pairs are behaving "nontrivially,"

MUST ENFORCE PARTICLE NUMBER CONSERVATION!