Cooper Pairing in "Exotic" Fermi Superfluids: An Alternative Approach

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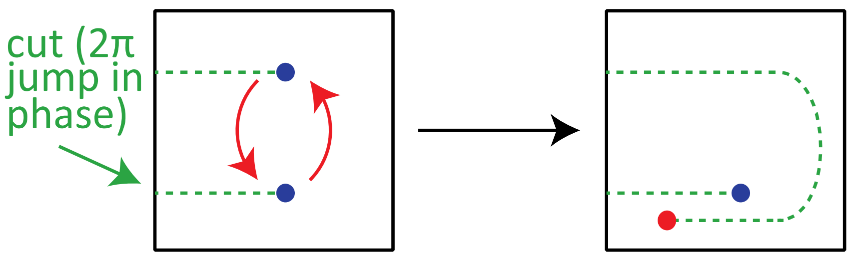
Why are MF's anyons?

Let's number the vortices 1, …, 2n in an arbitrary way, and consider the result of exchanging the vortices i and i + 1. Ivanov's* argt.: Vortex



$(\equiv \hat{T}_i)$

i "sees" no change in phase of superconducting order parameter $\Delta(\boldsymbol{r})$, vortex i + 1 sees a change of 2π. Hence

$$\hat{T}_i : \begin{cases} \hat{\gamma}_i \rightarrow \hat{\gamma}_{i+1} \\ \hat{\gamma}_{i+1} \rightarrow -\hat{\gamma}_i \\ \hat{\gamma}_j \rightarrow \hat{\gamma}_j \ \text{for } j \neq i, i+1 \end{cases}$$

note satisfies braid-group CR's.

*PRL **86**, 268 (2001).

Why are MF's anyons? (cont)

So: $$\hat{\tau}\left(\hat{T}_i\right) = \exp\left[\tfrac{\pi}{4}\hat{\gamma}_{i+1}\hat{\gamma}_i\right] = \tfrac{1}{\sqrt{2}}\left(1+\hat{\gamma}_{i+1}\hat{\gamma}_i\right)$$

(defined so that $\{\gamma'\} = \hat{\tau}(\hat{T}_i)\{\gamma\}\hat{\tau}^{-1}(\hat{T}_i)$)

Consider case of 2 fermions, then the Hilbert space is 2D, corresponding to no fermions present $\left(|0\rangle\right)$ and a single Dirac fermion, $a^\dagger|0\rangle \equiv |1\rangle$: this fermion is delocalized between the two vortices. In this basis we have

$$\hat{\tau}\left(\hat{T}\right) = \exp\left[i\tfrac{\pi}{4}\hat{\sigma}_z\right] \quad \left(\times \exp\left[-i\tfrac{\pi}{4}\hat{1}\right] \text{ for convenience}\right)$$

$$\equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

In words: exchanging the 2 vortices leaves state unchanged if no fermion present, changes phase by $\pi/2$ if one present.

Why are MF's anyons? (cont.)

qubit 1 qubit 2

The case of 2 fermions, "paired" as $1\leftrightarrow 2$, $3\leftrightarrow 4$:

$$\hat{\tau}\left(\hat{T}_1\right) = \exp\left[i\tfrac{\pi}{4}\hat{\sigma}_z^{(1)}\right]$$

$$\hat{\tau}\left(\hat{T}_3\right) = \exp\left[i\tfrac{\pi}{4}\hat{\sigma}_z^{(2)}\right]$$

but, what if we exchange 2 and 3: what is $\hat{\tau}\left(\hat{T}_2\right)$?
The form of $\hat{\tau}\left(\hat{T}_2\right)$ $(2\leftrightarrow 3)$ in the 4D Hilbert space:

Since $\gamma_3 \equiv a_2^\dagger + a_2,\ \gamma_2 \equiv i\left(a_1^\dagger - a_1\right)$

$$\hat{\tau}\left(\hat{T}_2\right) = \exp\left[\tfrac{\pi}{4}\hat{\gamma}_3\hat{\gamma}_2\right]$$

$$= \tfrac{1}{\sqrt{2}}(1+\gamma_3\gamma_2) = \tfrac{1}{\sqrt{2}}\left[1 + i\left(a_2^\dagger + a_2\right)\left(a_1^\dagger - a_1\right)\right]$$

$$= \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \equiv \tfrac{1}{\sqrt{2}}\left(1 + \hat{\sigma}_y^{(1)}\hat{\sigma}_x^{(2)}\right)$$

note suffixes refer to qubits not vortices!

entangling!

Why are MF's anyons? (cont.)

However, this is a bit misleading, because all operations preserve number parity of state. (as do all "real-life" physical operations). Hence, preferable to fix the parity and regard 4-anyon system as single qubit associated with e.g. anyons 1 and 2: *e.g*. for odd *N*

$$\left|0\right\rangle = \left|\text{no MF on } (1,2),\ \text{MF on } (3,4)\right\rangle$$
$$\left|1\right\rangle = \left|\text{MF on } (1,2),\ \text{no MF on } (3,4)\right\rangle$$

and for even *N*,

$$\left|0\right\rangle = \text{no MF's at all}$$
$$\left|1\right\rangle = \text{MF's on all 4 vortices}$$

With the above convention (for odd *N*),

$$\left(\left|0\right\rangle \equiv \left|(MF)_{12}, 0_{34}\right\rangle, \left|1\right\rangle \equiv \left|0_{12}, (MF)_{34}\right\rangle\right)$$

and we can verify that in both even and odd subspaces, $i\gamma_3\gamma_2 = \hat{\sigma}_x$.

Why are MF's anyons? (cont.)

The braiding matrices are (all × arb. overall abelian phase factors)

$$\hat{R}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \hat{R}_{34} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix},$$

$$\text{and } \hat{R}_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \exp -\mathrm{i}\frac{\pi}{4}\hat{\sigma}_x$$

i.e. multiplying R_{12}, R_{34} by $\exp \pm \mathrm{i}\pi/4$,

$$\hat{R}_{12} = \hat{R}_{34}^{-1} = \exp \mathrm{i}\tfrac{\pi}{4}\hat{\sigma}_z, \quad \hat{R}_{23} = \exp -\mathrm{i}\tfrac{\pi}{4}\hat{\sigma}_x$$

The R's so defined trivially satisfy the first braid-group relation,

$$\left[\hat{R}_i, \hat{R}_j\right] = 0 \text{ for } |i-j| \geq 2 \quad R_i \equiv R_{i,i+1}$$

Do they satisfy the second relation,

$$\hat{R}_i \hat{R}_{i+1} \hat{R}_i = \hat{R}_{i+1} \hat{R}_i \hat{R}_{i+1} \,?$$

Yes!

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \tfrac{1}{\sqrt{2}}\left(\hat{\sigma}_z - \hat{\sigma}_x\right) = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$$

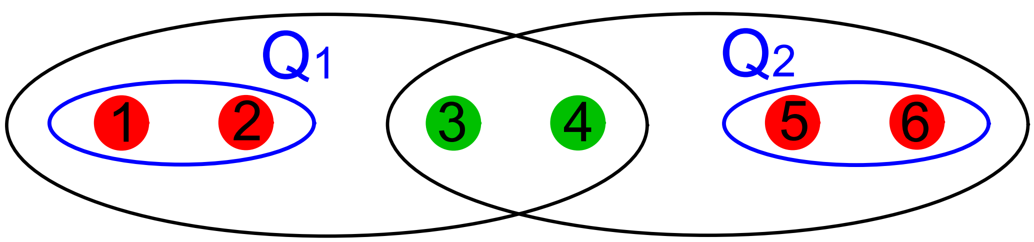
2 qubits

The simplest way of generating 2 qubits is to use $2n = 6$ anyons, and associate qubit 1 with the states of the anyon pair (1,2) and qubit 2 with pair (5,6). For definite, *e.g.* odd, parity the resulting Hilbert space is $2^{n-1} = 4$ dim'l, as it should be. The pair (3,4) is used, as before, to "soak up" any remaining parity. So, *e.g.*, we can take the 4 states to be, for odd N,

(1,2) (3,4) (5,6)

$$0_1 0_2 \equiv |0,1,0\rangle, \quad 1_1 0_2 \equiv |1,0,0\rangle,$$

$$0_1 1_2 \equiv |0,0,1\rangle, \quad 1_1 1_2 \equiv |1,1,1\rangle$$



What is the effect of braids $\hat{R}_{i,i+1}$?

As before,

$$\hat{R}_{12} = \exp\, i\tfrac{\pi}{4}\hat{\sigma}_z^{(1)} \text{ and sim. } \hat{R}_{56} = \exp\, i\tfrac{\pi}{4}\hat{\sigma}_z^{(2)}$$

2 qubits

Moreover, just as before

$$\hat{R}_{23} = \exp \text{-i}\tfrac{\pi}{4}\hat{\sigma}_x^{(1)} \text{ and sim. } \hat{R}_{45} = \exp \text{ i}\tfrac{\pi}{4}\hat{\sigma}_x^{(2)}$$

But what is the effect of $\hat{R}_{34}$? Acting on the "pseudoqubit" (3,4)

$$\hat{R}_{34} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} (\times \text{ abelian phase})$$

Hence,

$$\hat{R}_{34}|100\rangle = |100\rangle, \quad \hat{R}_{34}|001\rangle = |001\rangle,$$
$$\text{but } \hat{R}_{34}|010\rangle = i|010\rangle, \quad \hat{R}_{34}|111\rangle = i|111\rangle.$$

Thus in "2-qubit" basis$\left(|100\rangle \to |1_1 0_2\rangle\right)$, etc.

2-particle entangling gate!

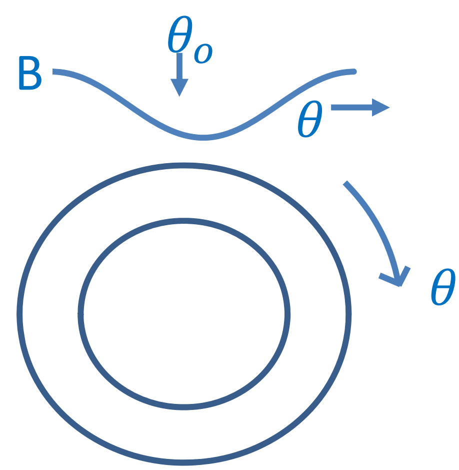
$$\hat{R}_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \equiv \exp \text{ i}\frac{\pi}{2}\left(\hat{\sigma}_{z1}\hat{\sigma}_{z2} + 1\right)$$

⇒ 6 anyons permit large set (but not complete one) of 1- and 2-qubit operations.

1. A cautionary tale

(combination of "SBU(1)S and Berry-phase arguments)

Consider a simple neutral s-wave BCS superfluid in a large ($R \gg \xi$) annulus. We construct a "Zeeman trap" s.t. the potential is $V(\theta) = -\mu\sigma_z B(\theta)$ with min. at θ_o: adjust parameters so that there is exactly one bound state of appropriate spin ($\uparrow$) in trap. "Width" of trap + hence of state $\gg \xi$., "shallow" so $E \sim \Delta$.



The $2N$ −particle GS Ψ_{2N} is clear: completely Cooper-paired state with COM at rest, effect of trap is $\sim(\mu B/\Delta)^2$, $L_z = O$ (so if trap is moved once around annulus, $\varphi_B = o$). In PNC approach, $2N$ is still defined mod 2, so same conclusion.

Now consider odd-parity GS. Presumably, CP's still at rest, one qp in trap., *i.e.* in PNC approach,

$$\Psi_{odd} = \int \big(u(r)\hat{\psi}^\dagger(r) + v(r)\hat{\psi}\,(r)\big)dr|\Psi_{even}\rangle \equiv \alpha_o^+|\Psi_{even}\rangle$$

In PC approach, (at first sight!)

$$\Psi_{2N+1} = \int \big(u(r)\hat{\psi}^\dagger(r) + v(r)\hat{\psi}(r)\hat{C}^\dagger\,\big)dr|\Psi_{2N}\rangle$$

In either case, relation to $2N$-particle CS there is an equal admixture of "extra particle" and "extra hole" in the trap region, so in that region there is an extra spin density but no extra particle density. There is also no current associated with the extra qp, hence no angular momentum and thus the Berry phase for "encirclement" by the trap is again zero.

Now suppose the condensate is moving, with the minimum velocity $h/2m$ (not h/m!) (as in case of Abrikosov vortex, $\xi \ll R \ll \lambda$). Again, we move the trap once around the annulus, adiabatically. The crucial question:

What is φ_B? (relative to the $2N$–particle circulating state: when presumably $\varphi = 2N\pi \sim o$)

(o, π, ill-defined, none of the above...)

(Note: in principle, experimentally meaningful question!)

A. Approach based on BdG equations

For the s-wave case and a spin-↑ qp, BDG equations read

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \mathrm{V}(r)\right)u(r) + \Delta(\boldsymbol{r})v(r) = Eu(r) \qquad \text{(etc.)}$$

where $\mathrm{v}(r) \equiv \mathrm{v}(\theta - \theta_o)$ and $\Delta(\boldsymbol{r}) \propto \exp i\theta$. For the moment, assume dependence on $\theta - \theta_o$ is "real", *i.e.* $u(\theta{:}\,\theta_o) = u(\theta_o)f(\theta - \theta_o), f$ real (etc.) (↑: not obvious!). Then this dependence cannot contribute to $\varphi_B \Rightarrow$ problem is exactly analogous to the "textbook" spin -1/2 problem with

$$\binom{\alpha}{\beta} \to \binom{u(\theta_o)}{v(\theta_o)} \qquad\qquad \varphi \to \theta_o$$

Moreover, for a bound state $|u|^2 = |v|^2 \Rightarrow$ equivalent to field in xy-plane

Hence by this analogy,

$\varphi_{B=\pi}$ (naïve BdG argument with SBU(1)S)

Note this also follows from the fact that $\ell_u - \ell_v = \ell_c = 1$

so $\langle L_z \rangle = \frac{1}{2}(\ell_u + \ell_v) = \frac{1}{2}(2\ell_u + 1) = n + 1/2$, n integer

B. Approach assuming PC:

Now, what if we try to modify this argument for the PC case? We now have

$$\alpha_o^+ = \int \{u(r)\psi^\dagger(r) + v(r)\psi(r)\hat{C}^\dagger\}$$

so while we still have $\ell_u - \ell_v = \ell_c = 1$, we now have

$$\langle L_z \rangle = \frac{1}{2}(\ell_u + \ell_v + \ell_c) = \frac{1}{2}(2\ell_v + 2) = \ell_v + 1 = \text{integer.}$$

Hence (mod 2π)

$$\varphi_B = 0 \qquad \text{(naïve argument, particle-conserving)}$$

However, let's revisit the dependence of $u(\theta - \theta_o)$ (etc.) on $\theta - \theta_o$. Does this contribute to the e.v. of the angular momentum $\langle L_z \rangle$ (*i.e.* is $i \int f^*(\theta - \theta_o) \frac{d}{d\theta} f(\theta - \theta_o) d\theta \neq 0$?)

In words:

when condensate is moving, is qp dragged with it or stationary with respect to the walls?

-- a surprisingly tricky question! (For stationary condensate, in complete absence of "normal" reflection, GS is 2-fold degenerate). Almost certainly qp dragged "partially", but not enough to give back BSBU(1)S – BdG result.

↑ amount depends on specifics of potential, etc.

Back to the Ivanov problem

According to Ivanov, if we have paired vortices j and $j+1$ and we interchange them, then

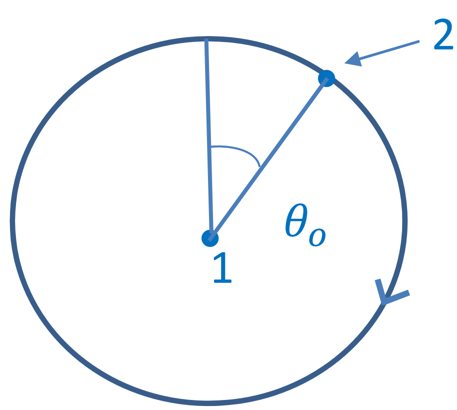
if no Majoranas, $\varphi_B = 0$

if two Majoranas, $\varphi_B = \pi/2$

It is more convenient to consider encirclement of $j(\text{at}\ \boldsymbol{r}=0)$ by $j+1$. So in our language, Ivanov's prediction for this operation is

even-number-parity states:
$\varphi_B = 0$

odd-number-parity states:
$\varphi_B = \pi$



Can we recover this prediction?

We use the fundamental result that since $\Psi_N(\theta_i) = \Psi_N\{\theta_i - \theta_o\}$,

$$\varphi_B = 2\pi\langle L_z\rangle.$$

(a) In the PNC approach (taken by Ivanov) for the odd-number-parity state

$$\langle L_z \rangle = \frac{1}{4}\left(\ell_{u_1} + \ell_{u_2} + \ell_{v_1} + \ell_{v_2}\right)$$

($\ell_{u1} \equiv$ ang. moment of "particle" component of Majorana in vortex 1, etc.)

However, since $u_1^*(r) = v_1(r), \ell_{u_1} = -\ell_{v_1}$ (etc.) $\Rightarrow \langle L_z \rangle = 0$. Thus, within PNC approach we have for the odd-number-parity state (mod. 2π)

$\varphi_B = 0$ different from Ivanov's result!

(b) In the PC approach, the $v-$ component of the Majorana is associated with creation of one extra Cooper pair. Hence

$$\langle L_z \rangle = \frac{1}{4}\left(\ell_{u_1} + \ell_{v_1} + \ell_c + \ell_{u_2} + \ell_{v_2} + \ell_c\right) = \frac{1}{2}\ell_c$$

where ℓ_c is the total angular momentum associated with the addition of an extra Cooper pair. This is just $\Omega_1 + \Omega_2 + \ell_{\text{int}}$

vortex 1 votex 2 relative a.m.

And since $\ell_{\text{int}} = 1$, and $\Omega_1 + \Omega_2 = 0$ or 2, ℓ_c is always an add integer and thus

$\varphi_B = \pi$ recovering Ivanov's result

↑: when we let (say) $u_1 \to v_1$ and thus add a Cooper pair, does that pair "feel" the effects of the angular momentum of the distant vortex 2?