INTRODUCTION TO TOPOLOGICAL QUANTUM COMPUTING

A. J. Leggett

Department of Physics
University of Illinois at Urbana-Champaign

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TOPOLOGICAL QUANTUM MEMORY/COMPUTING

Qubit basis. $|\uparrow\rangle, \ |\downarrow\rangle$

$$|\Psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

To preserve, need (for "resting" qubit)

$$\hat{H} \propto \hat{1} \qquad \text{in } |\uparrow\rangle, \ |\downarrow\rangle \text{ basis}$$

$$(\hat{H}_{12} = 0 \Rightarrow \ "T_1 \to \infty": \ \hat{H}_{11} = \hat{H}_{22} \Rightarrow "T_2 \to \infty")$$

on the other hand, to perform (single-qubit) operations, need to impose nontrivial $\hat{H}$.

⇒we must be able to do something Nature can't.

(ex: trapped ions: we have laser, Nature doesn't!)

Topological protection:

would like to find d–(>1) dimensional Hilbert space within which (in absence of intervention)

$$\hat{H} = (const.) \bullet \hat{1} + o\,(e^{-L/\xi})$$

size of system

microscopic length

How to find degeneracy?

Suppose ∃ two operators $\hat{\Omega}_1, \hat{\Omega}_2$ s.t.

$[\hat{H}, \hat{\Omega}_1] = [\hat{H}, \hat{\Omega}_2] = 0$ (and $\hat{\Omega}_1, \hat{\Omega}_2$ commutes with b.c's)

but

$[\hat{\Omega}_1, \hat{\Omega}_2] \neq 0$ (and $\hat{\Omega}_1 \,|\, \psi> \neq 0$)

EXAMPLE OF TOPOLOGICALLY PROTECTED STATE: FQH SYSTEM ON TORUS (Wen and Niu, PR B **41**, 9377 (1990))

Reminders regarding QHE:

2D system of electrons, $B \perp$ plane

Area per flux quantum = $(h/eB) \Rightarrow$ df.

$$\ell_M \equiv (\hbar / eB)^{1/2} \leftarrow \text{“magnetic length”}$$

$(\ell_M \sim 100 \dot{A}$ for B = 10 T)

“Filling fraction” ≡ no. of electrons/flux quantum ≡ ν

“FQH” when ν = p/q incommensurate integers

Argument for degeneracy: (does not need knowledge of w.f.)

can define operators of “magnetic translations”

$\hat{T}_x(\boldsymbol{a}), \hat{T}_y(\boldsymbol{b})$ (≡ translations of all electrons through **a**(**b**) × appropriate phase factors). In general $[\hat{T}_x(\boldsymbol{a}), \hat{T}_y(\boldsymbol{b})] \neq 0$

In particular, if we choose ← no. of flux quanta

$$\boldsymbol{a} = \boldsymbol{L}_1 / N_s, \;\; \boldsymbol{b} = \boldsymbol{L}_2 / N_s \qquad (= L_1 L_2 / 2\pi\ell^2)$$

then $\hat{T}_1, \hat{T}_2$ commute with b.c.'s and moreover

$$\hat{T}_1 \hat{T}_2 = \hat{T}_2 \hat{T}_1 \exp - 2\pi i \nu$$

But the o. of m. of $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\ell \cdot (\ell / L) \ll \ell$, and $\Rightarrow 0$ for $L \to \infty$.

Hence to a very good approximation,

$$[\hat{T}_1, \hat{H}] = [\hat{T}_2, \hat{H}] = 0 \qquad (*)$$

so since $[\hat{T}_1, \hat{T}_2] \neq 0$

must ∃ more than 1 GS (actually q).

Corrections to (*): suppose typical range of (e.g.) external potential V(**r**) is ℓ_o, then since $|\psi\rangle$'s oscillate on scale ℓ_{osc},

$$\langle \psi_1 | \hat{H} | \psi_2 \rangle \sim \exp - \ell_o / \ell_{osc} \sim \exp - L/\xi$$

$$(+\text{ const. } \hat{1})$$

Topological Protection and Anyons

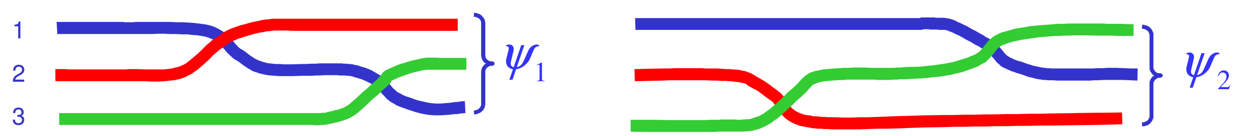
Anyons (df): exist only in 2D

$$\Psi(1,2) = \exp(2\pi i\alpha)\Psi(2,1) \equiv \hat{T}_{12}\Psi(1,2)$$

(bosons: α = 1, fermions: α = ½)

abelian if $\hat{T}_{12}\hat{T}_{23} = \hat{T}_{23}\hat{T}_{12}$ (ex: FQHE)

nonabelian if $\hat{T}_{12}\hat{T}_{23} \neq \hat{T}_{23}\hat{T}_{12}$, i.e., if



$$\psi_1 \neq \psi_2$$

("braiding statistics")

Nonabelian statistics* is a sufficient condition for topological protection:

[not necessary, cf. FQHE on torus]

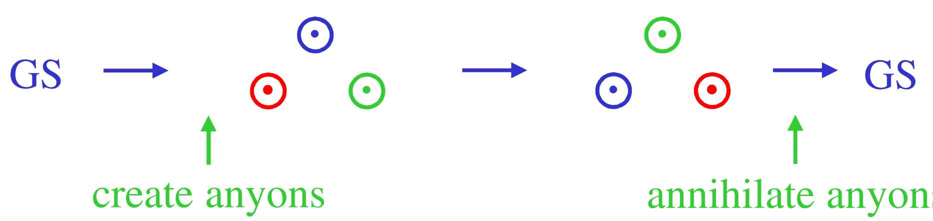
(a) state containing n anyons, $n \geq 3$:

$$[\hat{T}_{12}, \hat{H}] = [\hat{T}_{23}, \hat{H}] = 0$$

$$[\hat{T}_{12}, \hat{T}_{23}] \neq 0$$

$\Rightarrow$ space must be more than 1D.

(b) groundstate:



annihilation process inverse of creation $\Rightarrow$

GS also degenerate.

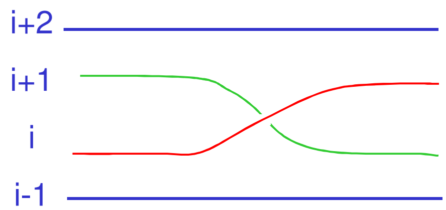
*plus gap for anyon creation

THE BRAID GROUP

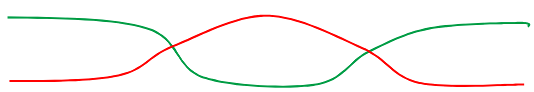
The braid group B_n for n objects is constructed by numbering Them arbitrarily 1, 2, …… n and defining

$$\hat{T}_i = \text{directed interchange of } i \text{ and } i+1.$$

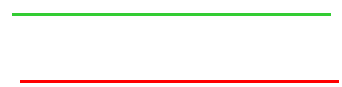
($\hat{T}_n$ = directed interchange of n and 1). (e.g. "i over $i+1$")



Note that $\hat{T}_i \neq \mathrm{P}_{i,i+1}$, and in particular we do not have $\hat{T}_i^2 = 1$:



$\neq$

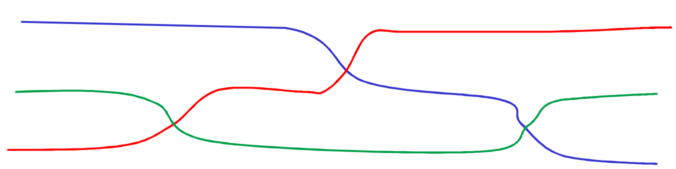


It is immediately obvious that the $\hat{T}_i$ satisfy

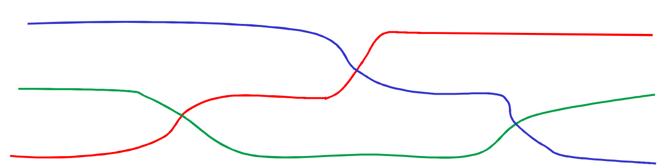
$$[\hat{T}_i, \hat{T}_j] = 0 \qquad \text{for } |i-j| > 1 \qquad (1)$$

Slightly less obviously,

$$\hat{T}_i \hat{T}_j \hat{T}_i = \hat{T}_j \hat{T}_i \hat{T}_j \qquad \text{for } |i-j| = 1 \qquad (2)$$



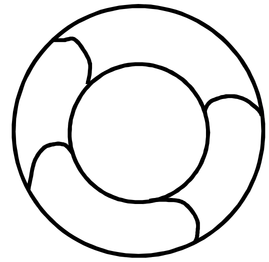
$=$



(1) and (2) may be taken as the definition of the braid group

SPECIFIC MODELS WITH TOPOLOGICAL PROTECTION

1. FQHE on torus



Obvious problems:

(a) QHE needs GaAs–AlGaAs or Si MOSFET: how to "bend" into toroidal geometry?

QHE observed in (planar) graphene (but not obviously "fractional"!): bend C nanotubes?

(b) Magnetic field should everywhere have large comp^{t} $\perp$ to surface: but div $\mathbf{B} = 0$ (Maxwell)!

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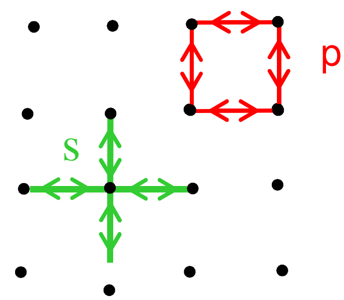
2. Spin Models (Kitaev et al.) (adv: exactly soluble)

(a) "Toric code" model

"Links" with spin ½ on lattice



$$\hat{H} = -\sum_s \hat{A}_s - \sum_p \hat{B}_p$$

$$\hat{A}_s \equiv \prod_{j \varepsilon s} \hat{\sigma}_j^x \,, \quad \hat{B}_p \equiv \prod_{j \varepsilon p} \hat{\sigma}_j^z$$

(so $[\hat{A}_s, \hat{B}_p] \neq 0$ in general)

Problems:

(a) toroidal geometry required (as in FQHE)

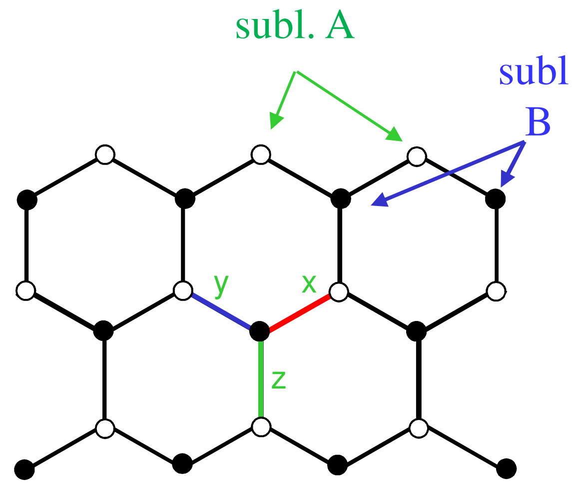
(b) apparently v. difficult to generate Ham^{n} physically

(c) does not permit topological quantum computation (only protection)

SPIN MODELS (cont.)

(b) Kitaev "honeycomb" model

Particles of spin ½ on honeycomb lattice (2 inequivalent sublattices, A and B)



$$\hat{H} = -J_x \sum_{x-links} \hat{\sigma}_j^x \hat{\sigma}_k^x - J_y \sum_{y-links} \hat{\sigma}_j^y \hat{\sigma}_k^y - J_z \sum_{z-links} \hat{\sigma}_j^z \hat{\sigma}_k^z$$

nb: spin and space axes independent

Strongly frustrated model, but exactly soluble.*

Sustains nonabelian anyons with gap provided

$$|J_x| \leq |J_y| + |J_z|, \; |J_y| \leq |J_z| + |J_x|,$$
$$|J_z| \leq |J_x| + |J_y| \qquad \text{and } \mathscr{H} \neq 0$$

magnetic field

(in opposite case anyons are abelian + gapped)

Advantages for implementation:

(a) plane geometry (with boundaries) is OK

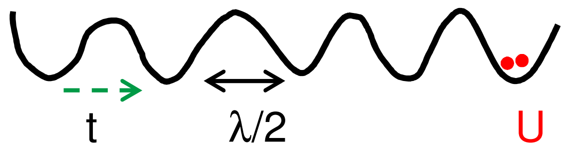
(b) $\hat{H}$ bilinear in nearest-neighbor spins

(c) permits partially protected TQC

* A. Yu Kitaev, Ann. Phys. 321,2 (2006)
H-D. Chen and Z. Nussinov, cond-mat/070363 (2007)

Some suggested implementations of the Kitaev honeycomb model:

(1) Optical lattices (Duan et al. PRL 91, 090402 (2003))



Can create 2D honeycomb lattice by suitable arrangements of lattices. By appropriate adjustments of laser polarizations, can make both t and U dependent on "spin" (hyperfine index). Then to 2nd order in t (Mott) get spin-dependent n.n. interaction.

Obvious problems:
(a) need for background trapping potential
(b) ultralong spin relax^{n}. times

(2) Polar molecules in optical lattices

(3) Josephson junction arrays

(4) Other solid-state lattice systems …

Problems: mostly massive "engineering"

So: different setup, occurring "naturally"?

The ν = 5/2 STATE

First seen in 1987: to date the only even-denom. FQHE state reliably established (some ev. for ν = 19/8). Quite robust: $\Sigma_{xy}/(e^2/h) = 5/2$ to high accuracy, excluding e.g. odd-denominator values ν = 32/13 or 33/13, and Σ_{xy} vanishes within exptl. accuracy. The gap $\Delta \sim 500$ mK.

WHAT IS IT?

If spin - polarized (probable), it is the $n = 1$ analog of ν = 1/2.

However, the actual ν = 1/2 state does not correspond to a FQHE plateau. In fact the CF (composite fermion) approach predicts that for this ν

$$N_\phi^{\ eff} = N_\phi - 2N_e = 0$$

and hence the CF's behave as a Fermi liquid: this seems to be consistent with expt. If LLL $\uparrow,\downarrow$ both filled, this argt. should apply equally to ν = 5/2 (since $(N_e/N_\phi)_{\mathrm{eff}} = ½$).

So what has gone wrong?

One obvious possibility †:

Cooper pairing of composite fermions!

since spins ǁ, must pair in odd – l state, e.g. p-state.

THE “PFAFFIAN” ANSATZ

Consider the Laughlin ansatz formally corresponding to $\nu = 1/2$:

$\psi^{(L)}{}_N = \prod_{i<j} (z_i - z_j)^2 \exp\text{-}\Sigma_I |z_i|^2/\text{f}l^2{}_m$ *($z_i \equiv x + iy$= electron coord.)*

This cannot be correct as it is symmetric under $i \rightleftarrows j$. So must multiply it by an antisymmetric function. On the other hand, do not want to “spoil” the exponent 2 in numerator, as this controls the relation between the LL states and the filling.

Inspired guess (Moore & Read, Greiter et. Al): (N = even)

$$\Psi^{(L)}{}_N = \Psi^{(L)} \times Pf\left(\frac{1}{z_i - z_j}\right)$$

Pf(f(ij)) ≡ *f(12)f(34)…-f(13)f(24). … + ….* (≡ Pfaffian)

↑

antisymmetric under ij

Is this consistent with expt.?

MOST PROMISING CANDIDATE FOR TQC IN "NATURAL" SYSTEMS: MAJORANA FERMIONS

System: 2D "$p + ip$" Fermi syperfluid
(? ^{3}He-A thin slabs, Sr_2RuO_4…?)

In such a system, half-quantum verticesmay occur. Near one of these, BdG (Bogoliubov–de Gennes) eqns possess a single solution with the properties

$$E = 0, \; u(\mathbf{r}) = v^*(\mathbf{r})$$

Such solultions are called Majorana fermions*

Crudely speaking,

2 Majorana fermions (on 2 separated vortices) $\cong$
1 "real" ("Dirac-Bogoliubov") fermion.

TQC can be achieved by "braiding" vortices with/without M.F.'s

*also believed to occur in s-wave superconductor next to topological insulator. Related excitations in $\nu = {}^5\!/_2$ QH state.

Brief Review of "Established Wisdom" on p + ip Fermi superfluids

order parameter

For a general spin-1/2 Fermi superfluid, OP df. by

$$F_{\alpha\beta}(\underset{\sim}{r},\underset{\sim}{r}') \equiv \left\langle \psi_\alpha^\dagger(r)\psi_\beta^\dagger(r')\right\rangle \leftarrow \text{"anomalous average"}$$

p + ip:

$$F_{\alpha\beta}(\underset{\sim}{r},\underset{\sim}{r}') = \delta_{\alpha\beta}F_\alpha(\underset{\sim}{r},\underset{\sim}{r}')$$

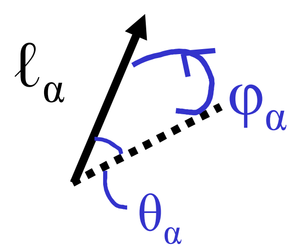
(ESP)
"equal spin pairing"

$$F_{\alpha\beta}(\underset{\sim}{r},\underset{\sim}{r}') \equiv F_\alpha(\underset{\sim}{R},\underset{\sim}{\rho}) \cong F_\alpha(\underset{\sim}{R})\,f_\alpha(\underset{\sim}{\rho})$$

COM ⟶ ⟵ rel. coord.

$$f_\alpha(\underset{\sim}{\rho}) \equiv \sin\theta_\alpha \exp i\varphi_\alpha$$

breaks TRI



thus if ℓ_α taken as z-axis, F. T. is

$$F_\alpha(\underset{\sim}{p}) = p_x + ip_y \;\leftarrow \text{"}p+ip\text{"}$$

Standard ansatz for MBWF (COM's of ↑, ↓ at rest):

$$\Psi = \Psi_\uparrow \Psi_\downarrow \qquad \text{(ESP)}$$

$$\Psi_\alpha \equiv \Big(\sum_{\boldsymbol{k}} c^\alpha_{\underset{\sim}{k}} a^+_{\underset{\sim}{k}\alpha} a^+_{-\underset{\sim}{k}\alpha}\Big)^{N/2} |\,\mathrm{vac}\rangle$$

where, if z-axis chosen along ℓ_α.

$$c^\alpha_k = f(|\underset{\sim}{k}|)\cdot(k_x + ik_y) \sim \exp i\varphi_{\underset{\sim}{k}} \qquad (p + ip)$$

Note dependence on φ_k extends to whole Fermi sea

Some properties of "standard" ansatz:

1. Ang. Momentum along $\ell_\alpha \cong N_\alpha \hbar / 2\big(-0(\Delta / E_F)\big)$ even in limit $\Delta \to 0$ or $T \to T_c$.
2. For 2D ("planar") case ($\underset{\sim}{\ell}_\alpha \perp$ plane), put $z = x + iy$, then for all $|z_i - z_j| \gg \xi$, coord-space MBWF is of form

   ξ: pair radius

$$\Psi_N(z_1 z_2 \ldots z_N) = \text{const.}\ \mathcal{A}\underbrace{\left\{\frac{1}{z_1 - z_2}\cdot\frac{1}{z_3 - z_4}\cdot\frac{1}{z_5 - z_6}\ldots.\right\}}_{\text{"Pfaffian"}}$$

Cf MR ansatz for $\nu = 5/2\ QH$ state

MR: Moore-Read

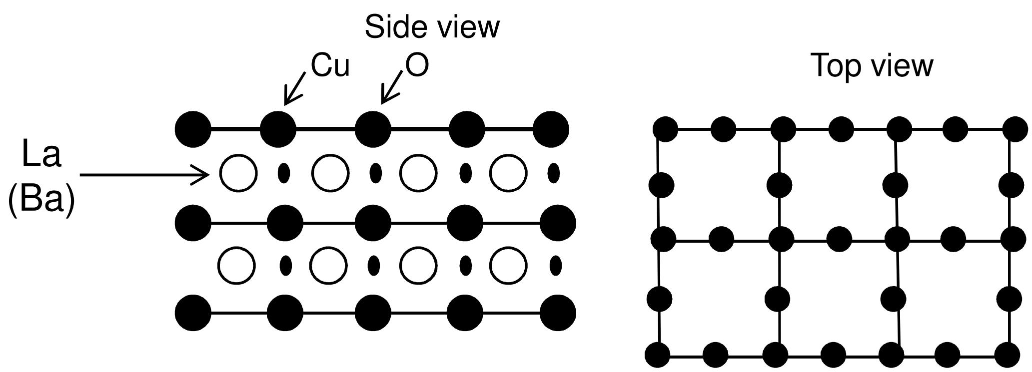
Strontium Ruthenate: Sr_2RuO_4

History

Superconductivity in cuprates up to ~ 150*K*

Typical (original) cuprate:

$La_{2-x}\,Ba_x\,CuO_4$ (T_C ~ 40*K*)

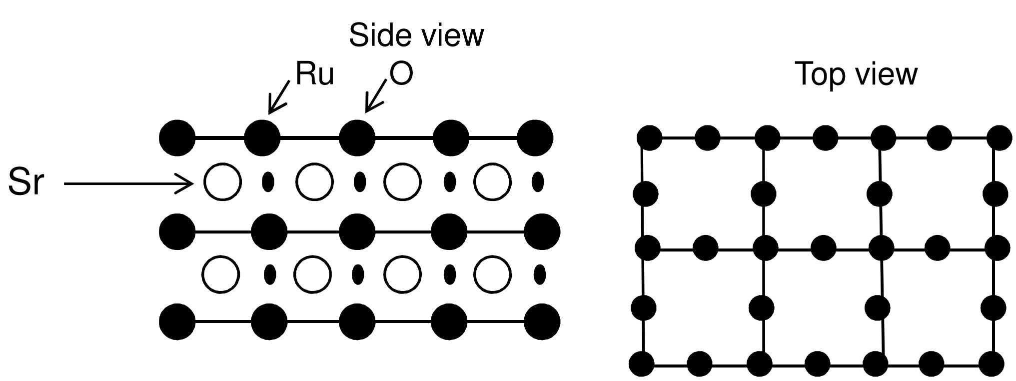


Quasi -2D CuO_2 planes appear to be essential to high – T_C superconductivity. How essential is the Cu? Try replacing it: Ag, Au…… doesn't work, but:

$$\text{Cu } (Z = 29) : \ [\text{Ar}\} + 3d^{10}4s^1 \ \rightarrow 3d^9$$
$$\text{Ru } (Z = 44) : \ [\text{Kr}] + 4d^7 5s^1 \rightarrow 4d^4$$

Normal-state props quite dissimilar

$La_{2-x}\,Ba_x\,CuO_4$ has T_C ~ 40*K*
Sr_2RuO_4 has T_C of ?



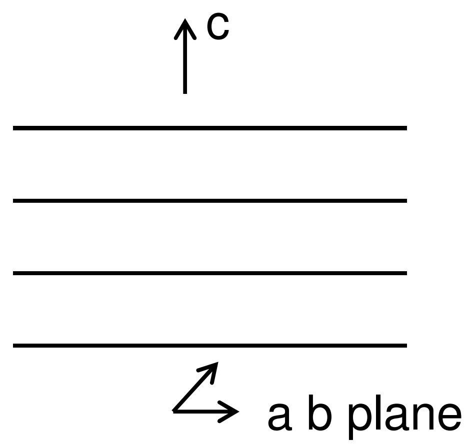
Experimental properties of Sr_2RuO_4*

Normal phase

Below ~25K, appears to behave as strongly anisotropic Fermi liquid (nb: cuprates quite different)

$C_V \sim \gamma T + \beta T^3$ $\qquad \chi \sim$ const.

electrons phonons



$\rho \sim A + BT^2$
both in ab-plane and $\rho \sim A + BT^2$ along c-axis (char. of coherent (Bloch wave) transport limited by $e^- - e^-$ Umklapp scattering). av. small (~1$\mu\Omega$ cm) →samples very pure.

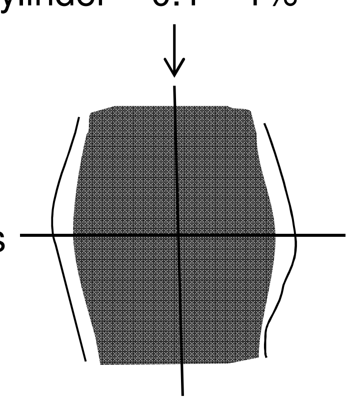
However, $\rho_c/\rho_{ab} \sim 10^3$ (comparable to cuprates)
Band structure:

Expt. (dHvA, Shubnikov-De Haas) and theory (LDA) agree: Fermi surface consists of 3 strongly 2D sheets : α (hole-like), β,γ (electron like)

k_F (Å^{-1})	0.3	0.6	0.75
m^*/m	3.3	7.0	16.0
m^*/m_b	3.0	3.5	5.5

← indication of strong correlations

Deviation from perfect cylinder ~ 0.1 – 1%



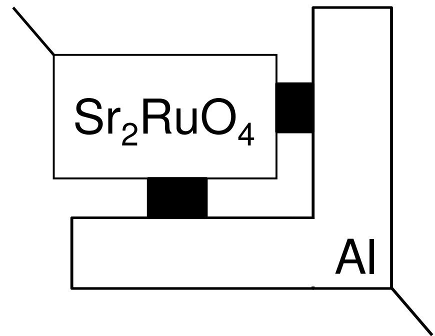
$\chi \sim$ const. in sup^{s} state $\Rightarrow$ triplet, ESP

* Mackenzie & Maeno, RMP **75**, 1 (2003)

Suppose OP of Sr2RuO4 is indeed of ESP form: then

$$F(\boldsymbol{k} : \sigma_1\sigma_2) = \quad F_\alpha(\boldsymbol{k}: \sigma_1\sigma_2) = \cdot f(\boldsymbol{k})\ (\uparrow\uparrow\ or\ \downarrow\downarrow)$$

Then the crucial question is:



What is f(***k***) ? ← orbital wf of pairs
In particular, is it real (e.g. $f(k) \sim k_x$)
or complex (e.g. $F(k) = k_x \pm ik_y$) ?
↑
breaks T – invariance

In BCS theory, want $|OP|^2$ to be as uniform as possible over F. surface → $k_x + ik_y$ always favored. But in more general theory, need not be so. Expts favoring violation of T:

a) Muon spin rotation extra "internal" magnetic field in superconductor state (↑ : app. Has ab-plane compt!)
b) Magnetic field dependence and telegraph noise in I_C of Josephson junctions interpreted in terms of switching of domains ($p_x + ip_y$ $px - ip_y$)
c) Kerr effect in zero magnetic field.

⇑: "ideal" $p_x + ip_y$ state ($\Delta \sim F \sim$ const. (k_x + ik) has no nodes

→ exponentially small no. of quasiparticles for $T \ll T_c$
→ no appreciable sp. ht., thermal conductivity…..

In fact, expt. evidence for power-law contribution of many of these quantities → gap has nodes?

In principle, critical test I_C max. at $\phi = ¼$ or $¾\ \phi_0$

Established Wisdom (cont.)

Half-quantum vortices ("HQV")

Should occur in any ESP Fermi superfluid, provided coupling between ↑↑ and ↓↓ sufficiently weak.

e.g. (neutral case):

vortex in ↑↑ components, nothing in ↓↓ component, i.e.

$\Delta_{\uparrow\uparrow} \propto \exp i\Phi$, $\Delta_{\downarrow\downarrow} \propto$ const. ("half-quantum" vortex)

Note, however, that quantization condition for ↑↑ pair velocity is still

$$K \equiv \int_0^{2\pi} \underset{\sim}{v}_{s\uparrow} \cdot d\underset{\sim}{l} = h/2m$$

Can tolerate Majorana fermions.

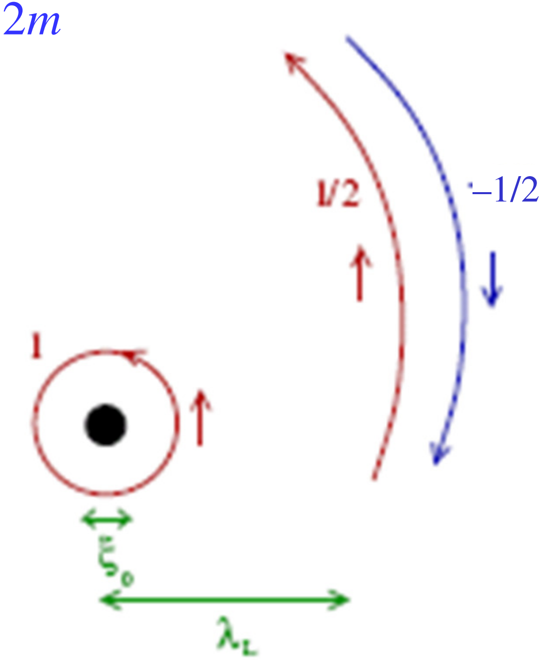
What about charged case?

At $r \ll \lambda_L$, current only in ↑↑ component → total $j \neq 0$.

However, for $r \gtrsim \lambda_L$, ↓↓'s are involved :

$$0 \neq \underset{\sim}{j}_{\uparrow} = \underset{\sim}{j}_{\downarrow}$$

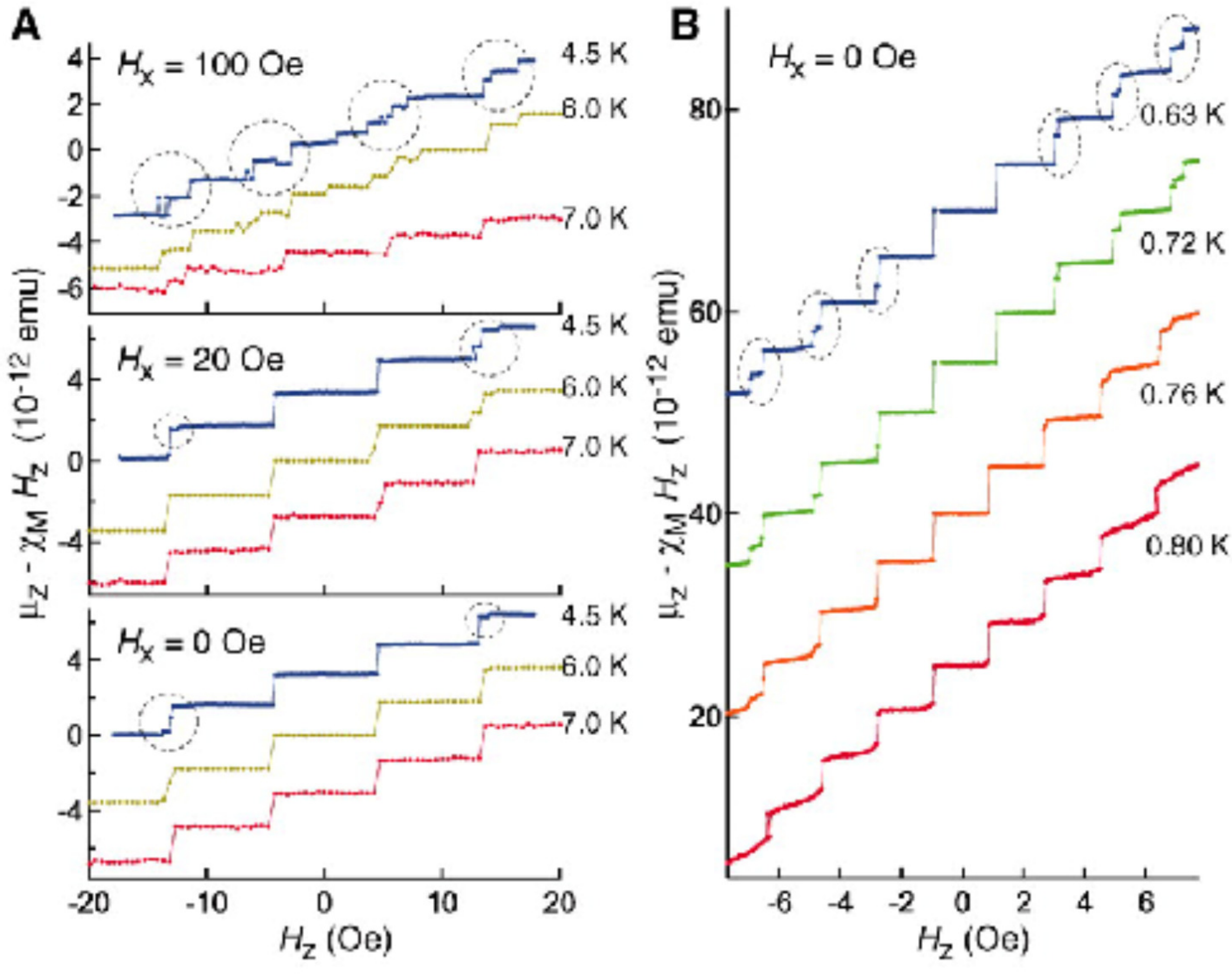
$$\underset{\sim}{j}_{tot} = 0$$



Total trapped flux = $\Phi_0/2 = nh/4e$.

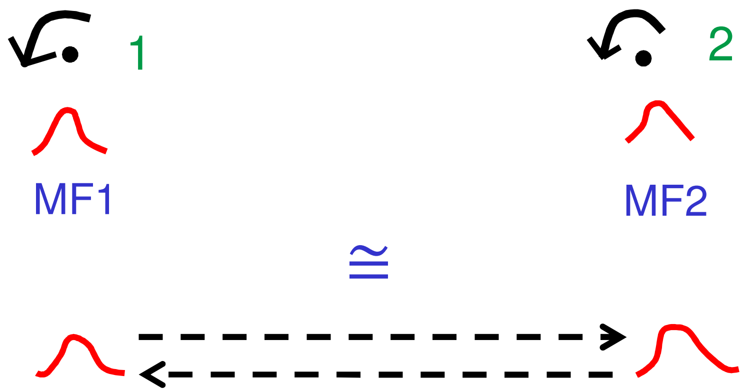
(Hence, vortex with Dirac (non-Majorana) fermion circling another picks up phase π/2, not π as for BCS)

Note: HQV in charged system carries circulating spin current as r→∞ ⇒ energetically disadvantaged relative to simple (h/2e) vortex.



J. Jang et al. Science 331, 186 (2011)

MAJORANA FERMIONS ON HQV's



single Dirac-Bogoliubov fermion with E=0

First important point:
impossible to tell, by any local meas^t. whether DC fermion is present or absent.

Second important point:
by appropriate braiding of vortices, can realize (for 2 vortices) nontrivial relative phase factors of the "occupied" and "unoccupied" states, and (for N>2 vortices) nonabelian operations.

For 2 vortices effect of 1 • ⇄ • 2 (interchange)

= effect of 2 • ↺ -1 (encirclement) <u>provided</u> we ignore

change of OP at position of 1, i.e. consider only $\Delta\psi_s = 2\pi$ for 2.

↑ change in phase of supercond *. OP

For a **<u>DB</u>** quasiparticle,

$\Delta\,\varphi = \frac{1}{2}\,\Delta\varphi_s = \pi$

For an M.F., only half is localized on 2 $\Rightarrow$
$\Delta\varphi = \frac{1}{4}\,\Delta\varphi_s = \pi/2$

MAJORANA FERMIONS ON HQV's

BRAIDING OF ANYONS

From the arguments of Ivanov*, if an E = 0 fermion is "shared" by 2 vortices and they are exchanged, MBWF changed by a factor of $\exp i\pi/2 \equiv i$, while if there is no fermion, this factor is 1. Hence, for the single-qubit system formed by 2 vortices

$$\hat{T}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad \left(or \exp i \, \frac{\pi}{4} \, \hat{\sigma}_z \right)$$

Case of 4 vortices:

At first sight, 2 qubits, e.g. associated with (1,2) and (3,4). $\Rightarrow$ 4D Hilbert space. Then:

$$\hat{T}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}_1 \hat{1}_2 \quad , \quad \hat{T}_{34} = \hat{1}_1 \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}_{34}$$

but

$$\hat{T}_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (1 + \hat{\sigma}_{y1} \, \hat{\sigma}_{x2}$$

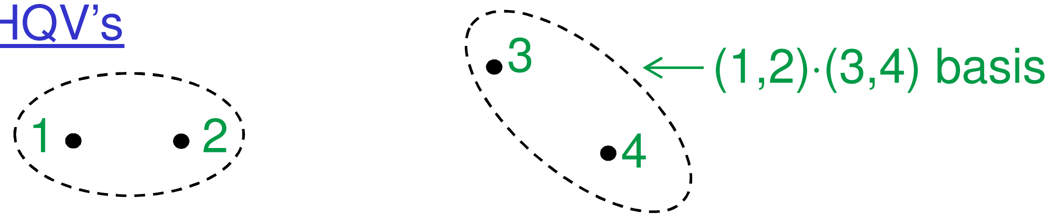
(entangling)

However this is a bit misleading, because all operations preserve parity of state. (as do all "real-life" physical operations). Hence, preferable to fix the parity and regard 4-anyon system as single qubit associated with e.g. anyons 1 and 2: e.g. for odd N

$|0\rangle$ = |no MF on (1,2), MF on (3,4)⟩
$|1\rangle$ = MF on (1,2), no MF on (3,4)⟩

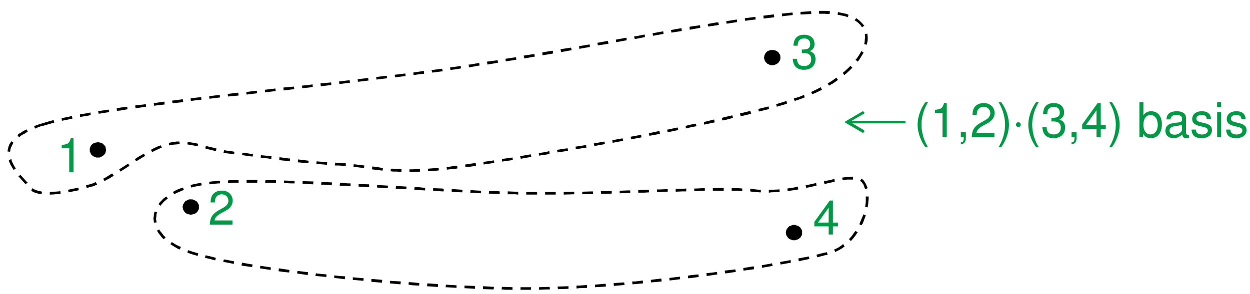
*PRL **86** (2001)

Case of 4 HQV's



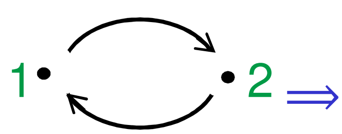
4-dim^{l}. E = 0 manifold, corresponding occupation or nonoccupation of (1,2) and (3,4)

But (claim!) can equally well pair differently, e.g.



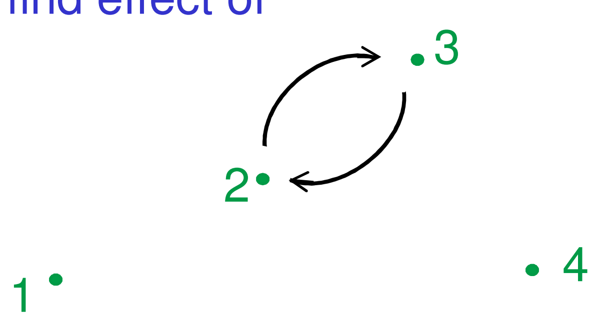
If the occupation states in this basis are single superpositions of those in original basis, then this just corresponds to a basic change in the 4D manifold.

We know this in original basis



$$\Rightarrow \begin{array}{cc} \begin{array}{cc}\text{occ} & \text{un}\end{array} & \\ \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} & \begin{array}{l}\text{occ}\\ \text{un}\end{array} \end{array}\Bigg|_{12} \; x \; \hat{1}_{34}$$

so to find effect of



write out in (13)=(24) basis and transform.