TOPOLOGICAL QUANTUM COMPUTING:

SOME POSSIBLY RELEVANT PHYSICAL SYSTEMS

A. J. Leggett

Department of Physics
University of Illinois at Urbana-Champaign

Topological Quantum Computing/Memory

Qubit basis. $|\uparrow\rangle, \ |\downarrow\rangle$

$$|\Psi\rangle = \alpha|\uparrow\rangle + \beta\,|\downarrow\rangle$$

To preserve, need (for "resting" qubit)

$\hat{H} \propto \hat{1}$ in $|\uparrow\rangle, \ |\downarrow\rangle$ basis

$$(\hat{H}_{12} = 0 \Rightarrow \ \text{"}T_1 \to \infty\text{"}: \ \hat{H}_{11} - \hat{H}_{22} = \text{const} \Rightarrow \text{"}T_2 \to \infty\text{"})$$

on the other hand, to perform (single-qubit) operations, need to impose nontrivial $\hat{H}$.

⇒ we must be able to do something Nature can't.

(ex: trapped ions: we have laser, Nature doesn't!)

Topological protection:

would like to find d–(>1) dimensional Hilbert space within which (in absence of intervention)

$$\hat{H} = (const.) \bullet \hat{1} + o\,(e^{-L/\xi})$$

size of system

microscopic length

How to find degeneracy?

Suppose ∃ two operators $\hat{\Omega}_1, \hat{\Omega}_2$ s.t.

$$[\hat{H}, \hat{\Omega}_1] = [\hat{H}, \hat{\Omega}_2] = 0 \quad (\text{and } \hat{\Omega}_1, \hat{\Omega}_2 \text{ commute with b.c's})$$

but

$$[\,\hat{\Omega}_1, \hat{\Omega}_2] \neq 0 \qquad (\text{and } \hat{\Omega}_1 \,|\, \psi> \neq 0)$$

then Hilbert space at least 2-dimensional…

EXAMPLE OF TOPOLOGICALLY PROTECTED STATE: FQH SYSTEM ON TORUS

(Wen and Niu, PR B **41**, 9377 (1990))

Reminders regarding QHE:

2D system of electrons, $B \perp$ plane

Area per flux quantum = $(h/eB) \Rightarrow$ df.

$$\ell_M \equiv (\hbar/eB)^{1/2}$$ ← “magnetic length”

$$(\ell_M \sim 100\dot{A} \text{ for B} = 10 \text{ T})$$

“Filling fraction” ≡ no. of electrons/flux quantum ≡ ν

“FQH” when ν = p/q   incommensurate integers

<u>Argument for degeneracy</u>: (does not need knowledge of w.f.)
can define operators of “magnetic translations”

$\hat{T}_x(\boldsymbol{a}), \hat{T}_y(\boldsymbol{b})$ (≡ translations of all electrons through **a**(**b**) × appropriate phase factors). In general $[\hat{T}_x(\boldsymbol{a}), \hat{T}_y(\boldsymbol{b})] \neq 0$

In particular, if we choose

no. of flux quanta $(= L_1 L_2 / 2\pi \ell_M^2)$ ↘

$$\boldsymbol{a} = \boldsymbol{L}_1 / N_s, \quad \boldsymbol{b} = \boldsymbol{L}_2 / N_s$$

then $\hat{T}_1, \hat{T}_2$ commute with b.c.'s (?) and moreover

$$\hat{T}_1 \hat{T}_2 = \hat{T}_2 \hat{T}_1 \exp - 2\pi i \nu$$

But the o. of m. of $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\ell_M \cdot (\ell_M / L) \equiv \ell_{osc} \ll \ell_M$, and ⇒ 0 for L→∞. Hence to a very good approximation,

$$[\hat{T}_1, \hat{H}] = [\hat{T}_2, \hat{H}] = 0 \qquad (*)$$

so since $[\hat{T}_1, \hat{T}_2] \neq 0$

must ∃ more than 1 GS (actually q).

Corrections to (*): suppose typical range of (e.g.) external potential $V(\mathbf{r})$ is ℓ_o, then since $|\psi\rangle$'s oscillate on scale ℓ_{osc},

$$\langle \psi_1 | \hat{H} | \psi_2 \rangle \sim \exp - \ell_o / \ell_{osc} \sim \exp - L / \underset{\uparrow}{\xi}$$

(+ const. $\hat{1}$)

$$\equiv \ell_M^2 / \ell_o$$

TOPOLOGICAL PROTECTION AND ANYONS

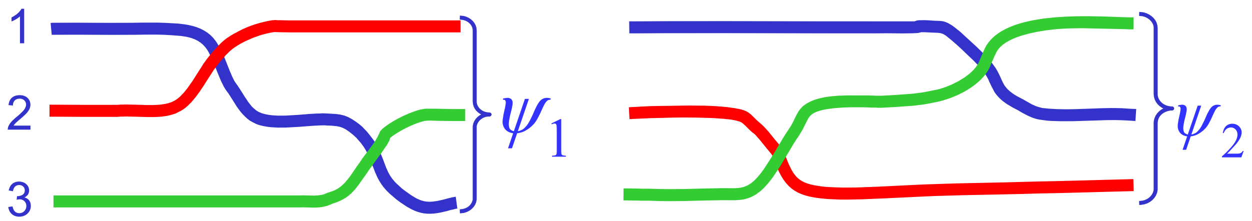
Anyons (df): exist only in 2D

$$\Psi(1,2) = \exp(2\pi i\alpha)\Psi(2,1) \equiv \hat{T}_{12}\Psi(1,2)$$

(bosons: $\alpha = 1$, fermions: $\alpha = ½$)

abelian if $\hat{T}_{12}\hat{T}_{23} = \hat{T}_{23}\hat{T}_{12}$ (ex: FQHE)

nonabelian if $\hat{T}_{12}\hat{T}_{23} \neq \hat{T}_{23}\hat{T}_{12}$, i.e. if



$$\psi_1 \neq \psi_2$$

("braiding statistics")

Nonabelian statistics* is a sufficient condition for (partial) topological protection:

[not necessary, cf. FQHE on torus]

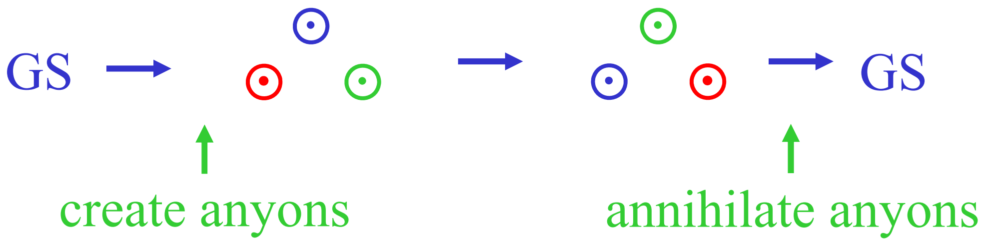
(a) state containing *n* anyons, $n \geq 3$:

$$[\hat{T}_{12},\hat{H}]=[\hat{T}_{23},\hat{H}]=0$$

$$[\hat{T}_{12},\hat{T}_{23}]\neq 0$$

$\Rightarrow$ space must be more than 1D.

(b) groundstate:



annihilation process inverse of creation $\Rightarrow$

GS also degenerate.

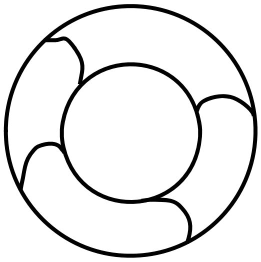
*plus gap for anyon creation

Nonabelian statistics may (depending on type) be adequate for (partially or wholly) topologically protected quantum computation

SPECIFIC MODELS WITH TOPOLOGICAL PROTECTION

1. FQHE on torus

Obvious problems:



(a) QHE needs GaAs–AlGaAs or Si MOSFET: how to "bend" into toroidal geometry?

QHE observed in (planar) graphene (but not obviously "fractional"!): bend C nanotubes?

(b) Magnetic field should everywhere have large compt ⊥ to surface: but div **B** = 0 (Maxwell)!

(c) anyway, anyons are Abelian, so permits only topological protection (not TP computation)

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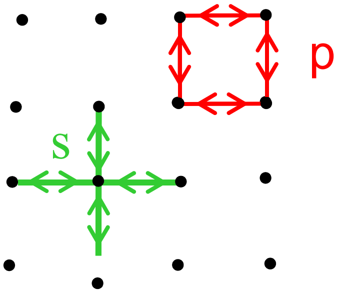
2. Spin Models (Kitaev et al.) (adv: exactly soluble)

(a) "Toric code" model

Particles of spin ½ on lattice



$$\hat{H} = -\sum_s \hat{A}_s - \sum_p \hat{B}_p$$

$$\hat{A}_s \equiv \prod_{j\varepsilon s} \hat{\sigma}_j^x \,, \quad \hat{B}_p \equiv \prod_{j\varepsilon p} \hat{\sigma}_j^z$$

$$\text{(so } [\hat{A}_s, \hat{B}_p] \neq 0 \text{ in general)}$$

Problems:

(a) in original formulation, toroidal geometry required (as in FQHE)

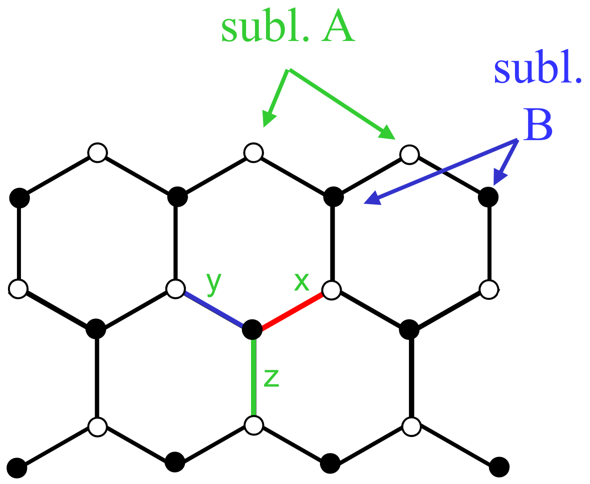
(b) apparently v. difficult to generate Hamⁿ physically

however, developments of this idea ⇒ surface codes using Josephson qubits

SPIN MODELS (cont.)

(b) Kitaev "honeycomb" model

Particles of spin ½ on honeycomb lattice (2 inequivalent sublattices, A and B)



$$\hat{H} = -J_x \sum_{x-links} \hat{\sigma}_j^x \hat{\sigma}_k^x - J_y \sum_{y-links} \hat{\sigma}_j^y \hat{\sigma}_k^y - J_z \sum_{z-links} \hat{\sigma}_j^z \hat{\sigma}_k^z - \mathscr{H} \cdot \sum_{sites} \boldsymbol{\sigma}$$

nb: spin and space axes independent

Strongly frustrated model, but exactly soluble.*

Sustains nonabelian anyons with gap provided

$$|J_x| \le |J_y| + |J_z|, \quad |J_y| \le |J_z| + |J_x|,$$
$$|J_z| \le |J_x| + |J_y| \qquad \text{and } \mathscr{H} \neq 0$$

(in opposite case anyons are abelian + gapped)

Advantages for implementation:

(a) plane geometry (with boundaries) is OK

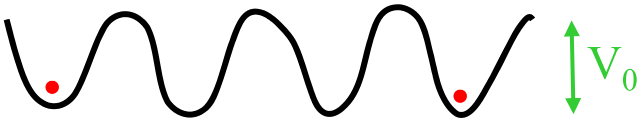
(b) $\hat{H}$ bilinear in nearest-neighbor spins

(c) permits partially protected quantum computation.

* A. Yu Kitaev, Ann. Phys. 321, 2 (2006)
H-D. Chen and Z. Nussinov, cond-mat/070363 (2007)
(etc. …)

Can we Implement Kitaev Honeycomb Model?

One proposal (Duan et al., PRL *91*, 090492 (2003)): use optical lattice to trap ultracold atoms



Optical lattice:

3 counterpropagating pairs of laser beams create potential, e.g. of form

(2π/λ laser wavelength)

$$V(\boldsymbol{r}) = V_o(\cos^2 kx + \cos^2 ky + \cos^2 kz)$$

in 2D, 3 counterpropagating beams at 120° can create honeycomb lattice (suppress tunnelling along z by high barrier)

For atoms of given species (e.g. ^{87}Rb) in optical lattice 2 characteristic energies:

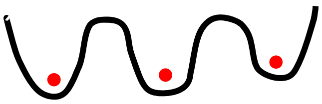
interwell tunnelling, t $(\sim e^{-\text{const.}\sqrt{V_0}})$

intrawell atomic interaction (usu. repulsion) U

For 1 atom per site on average:

if t » U, mobile (“superfluid”) phase

if t « U, “Mott-insulator”phase (1 atom localized on each site)



If 2 hyperfine species (≅ “spin –1/2” particle), weak intersite tunnelling ⇒ AF interaction

$$\hat{H}_{AF} = \sum_{nn} J\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j \qquad J = t^2/U$$

(irrespective of lattice symmetry).

So far, isotropic, so not Kitaev model. But …

If tunnelling is different for ↑ and ↓, then H’berg Hamiltonian is anisotropic: for fermions,

$$\hat{H}_{AF} = \frac{t_\uparrow^2 + t_\downarrow^2}{2U} \sum_{nn} \hat{\sigma}_i^z \hat{\sigma}_j^z + \frac{t_\uparrow t_\downarrow}{U} \sum_{nn} (\hat{\sigma}_i^x \hat{\sigma}_j^x + \hat{\sigma}_i^y \hat{\sigma}_j^y)$$

$\Rightarrow$ if $t_\uparrow \gg_\downarrow$, get Ising-type int^n

$$H_{AF} = \text{const.} \sum_{nn} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

We can control $t_\uparrow$ and $t_\downarrow$ with respect to an arbitrary “z” axis by appropriate polarization and tuning of (extra) laser pair. So, with 3 extra laser pairs polarized in mutually orthogonal directions (+ appropriately directed) can implement

$$\hat{H} = J_x \sum_{x-bonds} \hat{\sigma}_i^x \hat{\sigma}_j^x + J_y \sum_{y-bonds} \hat{\sigma}_i^y \hat{\sigma}_j^y + J_z \sum_{z-bonds} \hat{\sigma}_i^z \hat{\sigma}_j^z$$

≡ Kitaev honeycomb model

Some potential problems with optical-lattice implementation:

(1) In real life, lattice sites are inequivalent because of background magnetic trap ⇒ region of Mott insulator limited, surrounded by “superfluid” phase.

(2) to avoid thermal excitation, need $T \lesssim =$ 1pK. (10^{-12}K)

(3) Even if “T” < 1pK, v. long “spin” relaxation times in ultracold atomic gases ⇒ true groundstate possibly never reached.

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Other possible implementations: e.g. Josephson circuits (You et al., arXiv: 0809.0051)

QUANTUM HALL SYSTEMS

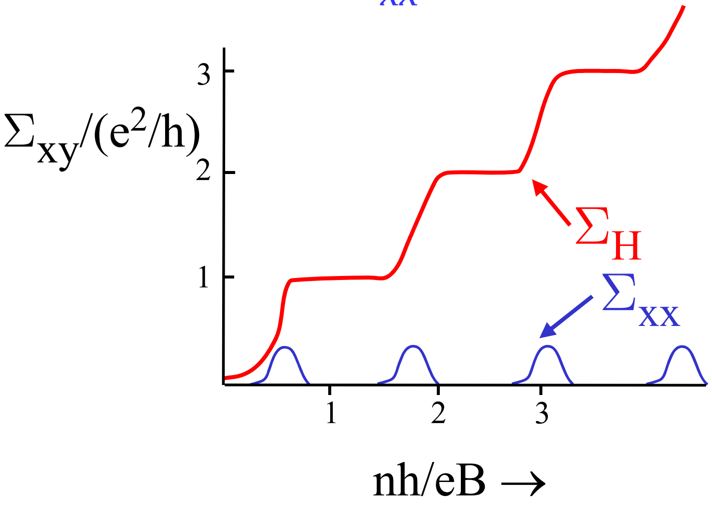
Reminder re QHE:

Occurs in (effectively) 2D electron system (“2DES”)
(e.g. inversion layer in GaAs – GaAlAs heterostructure)
in strong perpendicular magnetic field, under conditions of high purity and low ($\lesssim$ 250 mK) temperature.

If df. $l_m \equiv (\hbar/eB)^{1/2}$ (“magnetic length”) then area per flux quantum h/e is $2\pi l_m^2$, so no. of flux quanta $= A/2\pi l_m^2$ ($A \equiv$ area of sample). If total no. of electrons is N_e, define

$$\nu \equiv N_e / N_\Phi \qquad \text{(“filling factor”)}$$

QHE occurs at and around (a) integral values of ν (integral QHE) and (b) fractional values p/q with fairly small ($\lesssim$ 13) values of q (fractional QHE). At ν’th step, Hall conductance Σ_{xy} quantized to $\nu e^2/\hbar$ and longitudinal conductance $\Sigma_{xx} \cong 0$



Nb: (1) Fig. shows IQHE only

(2) expts usually plot

$$R_{xy} \text{ vs } B \left(\propto \frac{1}{\nu} \right)$$

so general pattern is same but details different

ν = 5/2 STATE: THE "PFAFFIAN" ANSATZ

Consider the Laughlin ansatz formally corresponding to ν = 1/2 (or ν = 5/2 with first 2 LC's inert):

$$\psi_N^L = \Pi_{i<j}(z_i - z_j)^2 \exp - \Sigma_i |z_i|^2 /4l_m^2 \quad (z_i = \text{electron coord.})$$

This cannot be correct as it is symmetric under $i \leftrightarrow j$. So must multiply it by an antisymmetric function. On the other hand, do not want to "spoil" the exponent 2 in numerator, as this controls the relation between the LL states and the filling.

Inspired guess (Moore & Read, Greiter et al.): (N = even)

$$\psi_N = \psi_N^{(L)} \times Pf\left(\frac{1}{z_i - z_j}\right)$$

$$Pf(\underset{\substack{\uparrow \\ \text{antisymmetric under } ij}}{f(ij)}) \equiv f(12)f(34)... - f(13)f(24).... + ...(\equiv \text{ Pfaffian})$$

This state is the exact GS of a certain (not very realistic) 3-body Hamiltonian, and appears (from numerical work) to be not a bad approximation to the GS of some relatively realistic Hamiltonians.

With this GS, a single quasihole is postulated to be created, just as in the Laughlin state, by the operation

$$\psi_{qh} = \left(\Pi_{i=1}^N (z_i - \eta_0)\right) \bullet \psi_N$$

It is routinely stated in the literature that "the charge of a quasihole is –*e*/4", but this does not seem easy to demonstrate directly: the arguments are usually based on the BCS analogy (quasihold ↔ *h/2e* vortex, extra factor of 2 from usual Laughlin-like considerations) or from CFT.

conformal field theory

These excitations are nonabelian ("Ising") anyons. ⇒ permit partially protected quantum computation.

p-WAVE FERMI SUPERFLUIDS (in 2D)

Generically, particle-conserving wave function of a Fermi superfluid (Cooper-paired system) is of form

$$\Psi_N = \mathcal{N} \cdot \Big(\sum_{k,\alpha\beta} c_k a^+_{k\alpha} a^+_{-k\beta} \Big)^{N/2} |vac\rangle$$

e.g. in BCS superconductor

$$\Psi_N = \mathcal{N} \Big(\sum_{k} c_k a^+_{k\uparrow} a^+_{-k\downarrow} \Big)^{N/2} |vac\rangle -$$

Consider the case of pairing in a spin triplet, p-wave state (e.g. 3He-A). If we neglect coherence between ↑ and ↓ spins, can write

$$\Psi_N = \Psi_{N/2,\uparrow} \Psi_{N/2,\downarrow}$$

Concentrate on $\Psi_{N/2,\uparrow}$ and redef. N→ 2N.

$$\Psi_{N\uparrow} = \mathcal{N} (\Sigma c_k a^+_k a^+_{-k})^{N/2} |vac\rangle$$

suppress spin index

What is c_k?

Standard choice:

KE measured from μ

$$c_k = \exp - i\phi_k \left(\frac{1-\varepsilon_k / E_k}{1+\varepsilon_k / E_k}\right)^{1/2}$$

Real factor

“p+ip”

$$\left(\varepsilon_k^2 + |\Delta_k|^2\right)^{1/2}$$

How does c_k behave for k→0?

For p-wave symmetry, $|\Delta_k|$ must $\propto$ k, so

$$|c_k| \sim \varepsilon_F / |\Delta_k| \sim k^{-1}$$

Thus the (2D) Fournier transform of c_k is

$$\propto r^{-1} \exp - i\varphi \equiv z^{-1},$$

and the MBWF has the form

$$\Psi_N(z_1 z_2 \ldots z_N) = Pf\left(\frac{1}{z_i - z_j}\right) \times \text{ uninteresting factors}$$

Conclusion: apart from the “single-particle” factor $\exp - \frac{1}{4\ell^2}\sum_j |z_j|^2$, the “standard” real-space MBWF of a (p + ip) 2D Fermi superfluid is identical to the MR ansatz for ν = 5/2 QHE. Note one feature of the latter:

if $\hat{\Omega} \equiv \sum_k c_k a_k^+ a_{-k}^+ , \quad c_k = |c_k| \exp - i\varphi_k$

then $[\hat{L}_{z,} \hat{\Omega}] = -\hbar\hat{\Omega}$

z-component of ang. momentum

so $\Psi_N \equiv \text{const.}\ \hat{\Omega}^N |vac\rangle$

possesses ang. momentum –N$\hbar$/2, no matter how weak the pairing!

Now: where are the nonabelian anyons in the p + ip Fermi superfluid?

Read and Green (Phys. Rev. B *61*, 10217(2000)): nonabelian anyons are zero-energy fermions bound to cores of vortices.

Consider for the moment a single-component 2D Fermi superfluid, with $p + ip$ pairing. Just like a BCS (s-wave) superconductor, it can sustain vortices: near a vortex the pair wf, or equivalently the gap $\Delta(\mathbf{R})$, is given by

COM of Cooper pairs

$$\Delta(\boldsymbol{R}) \equiv \Delta(z) = \text{ const. } z$$

Since $|\Delta\,(\mathbf{R})|^2 \to 0$ for $\mathbf{R} \to 0$, and (crudely) $E_k(\boldsymbol{R}) \sim (\varepsilon_k^2 + |\Delta(\boldsymbol{R})|^2)^{1/2}$, bound states can exist in core.

In the s-wave case their energy is $\sim\eta\,|\Delta_o|^2\,\varepsilon_F$, $\eta \neq 0$, so no zero-energy bound states.

What about the case of $(p + ip)$ pairing?

If we approximate

$$\Delta(\boldsymbol{R},\boldsymbol{\rho}) = \Delta(R)\partial_\rho\delta(\boldsymbol{\rho})$$

relative coord.

$\exists$ mode with $u(\mathbf{r}) = v^*(\mathbf{r})$, $E = 0$

Now, recall that in general within mean-field (BdG) theory,

$$\psi_{odd}(\mathbf{r}) = \left(u(r)\hat{\psi}^{\dagger}(r) + \upsilon(r)\hat{\psi}(r)\right)|GS\rangle \equiv \hat{Q}(r)|GS\rangle$$

But, if $u^*(r) = \upsilon(r)$, then $\hat{Q}^{\dagger}(r) \equiv \hat{Q}(r)$! i.e.

zero-energy modes are their own antiparticles
("Majorana modes")

⇒ undetectable by any local probe

⚠: This is true only for spinless particle/pairing of || spins (for pairing of anti || spins, particle and hole distinguished by spin).

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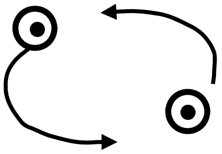
* Ivanov, PRL *86*, 268 (2001)

Consider two vortices i, j with attached Majorana modes with creation ops. $\gamma_i \equiv \gamma_i^\dagger$.



What happens if two vortices are interchanged?*

Claim: when phase of C. pairs changes by 2π, phase of Majorana mode changes by π (true for assumed form of u, v for single vortex). So

$$\gamma_i \to \gamma_j$$

$$\gamma_j \to -\gamma_i$$

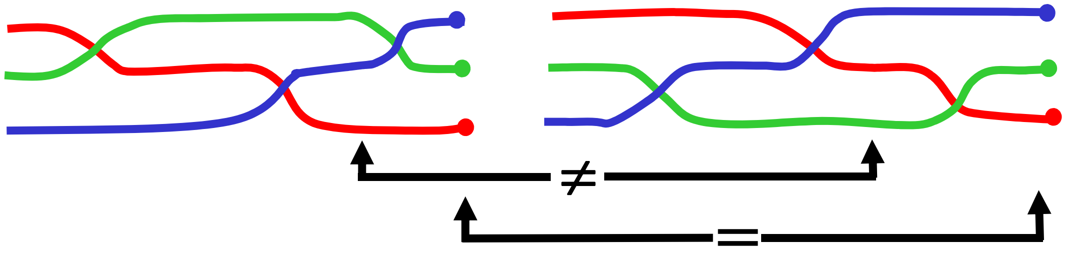
more generally, if ∃ many vortices + w df $\hat{T}_i$ as exchanging $i, i+1$, then for $|i–j|>1$

$$[\hat{T}_i, \hat{T}_j] = 0, \text{ but}$$

for $|i-j|=1$, $[\hat{T}_i, \hat{T}_j] \neq 0$, $\hat{T}_i \hat{T}_j \hat{T}_i = \hat{T}_j \hat{T}_i \hat{T}_j$

⇒ Majoranas are Ising anyons

braid group!



* Ivanov, PRL *86*, 268 (2001)

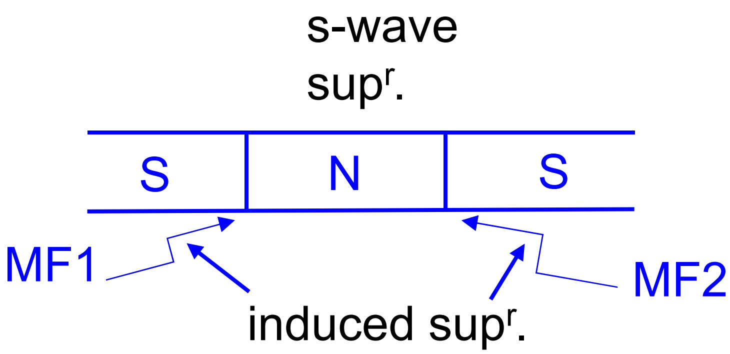
The experimental situation

Sr_2RuO_4: so far, evidence for HQV's, none for MF's.

^{3}He-A: evidence if anything against HQV's

^{3}He-B: circumstantial evidence from ultrasound attenuation

Alternative proposed setup (very schematic)



s-wave
sup^r.

S N S

MF1

MF2

induced sup^r.

zero-bias anomaly

Detection: ZBA in I-V characteristics

(Mourik et al., 2012, and several subsequent experiments)

dependence on magnetic field, s-wave gap, temperature... roughly right

"What else could it be?"

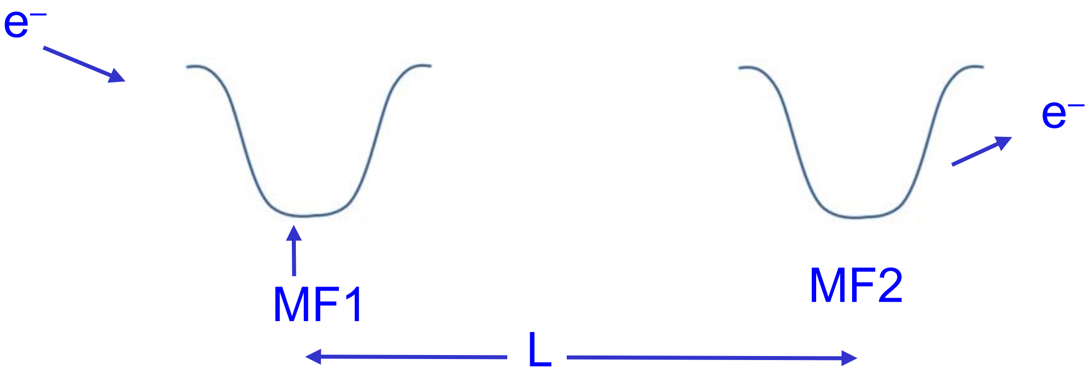
Answer: quite a few things!

Second possibility: Josephson circuit involving induced (p-wave-like) sup^{y}.

Theoretical prediction: “4π-periodicity” in current-phase relation.

Problem: parasitic one-particle effects can mimic.

One possible smoking gun: teleportation!



$\Delta \mathrm{T} \ll L/v_F$?

Fermi velocity

Problem: theorists can’t agree on whether teleportation is for real!

Majorana fermions: beyond the mean-field approach

Problem: The whole apparatus of mean-field theory rests fundamentally on the notion of SBU(1)S ← spontaneously broken U(1) gauge symmetry:

$$\Psi_{\text{even}} \sim \sum_{\substack{N= \\ \text{even}}} C_N \, \Psi_N \qquad \left(C_N \sim |C_N| e^{iN\varphi}\right)$$

$$\Psi_{\text{odd}}^{(c)} \sim \int dr \left\{u(r)\hat{\psi}^{\dagger}(r) + v(r)\hat{\psi}(r)\right\}|\Psi_{\text{even}}\rangle \left(\equiv \hat{\gamma}_i^{\dagger}|\Psi_{\text{even}}\rangle\right) *$$

But in real life condensed-matter physics,

SB U(1)S IS A MYTH!!

This doesn’t matter for the even-parity GS, because of “Anderson trick”:

$$\Psi_{2\text{N}} \sim \int \Psi_{\text{even}}(\varphi) \exp -iN\varphi \, d\varphi$$

But for odd-parity states equation (*) is fatal! Examples:

(1) Galilean invariance

(2) NMR of surface MF in ^{3}He-B

We must replace (*) by

creates extra Cooper pair

$$\hat{\gamma}_i^\dagger = \int dr \left\{ u(r)\hat{\psi}^\dagger(r) + v(r)\hat{\psi}\overset{\downarrow}{C^\dagger} \right\}$$

This doesn't matter, so long as Cooper pairs have no "interesting" properties (momentum, angular momentum, partial localization...)

But to generate MF's, pairs must have "interesting" properties!

⇒ doesn't change arguments about existence of MF's, but completely changes arguments about their braiding, undetectability etc.

Need completely new approach!