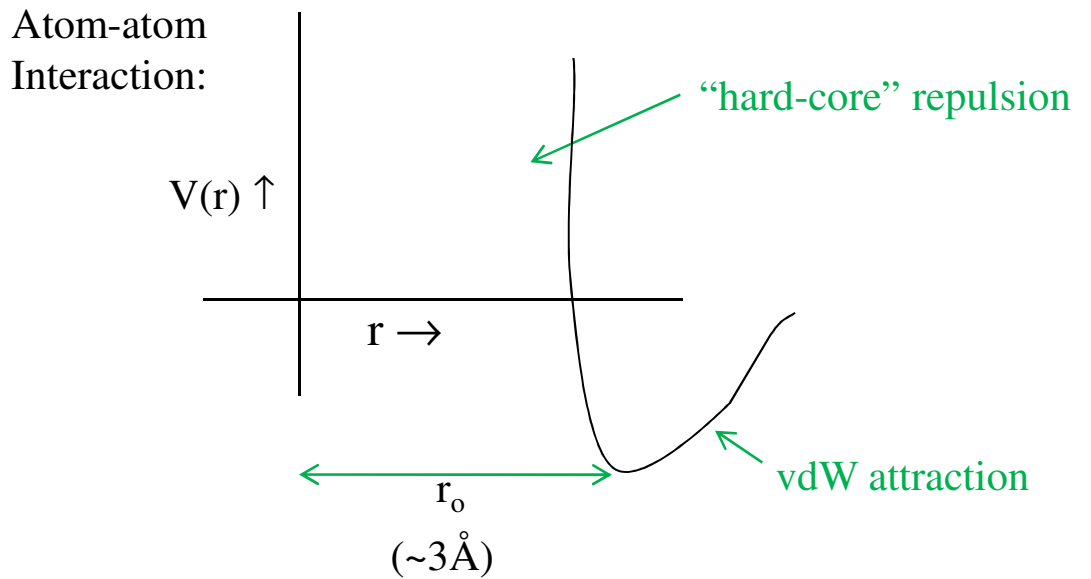


Lecture 2 – Superfluid ^3He : Basic Description

Ref: AJL, Quantum Liquids, 6.1-2

^3He : electronic state $(1s)^2 \ ^1S_0$, is very inert. Nuclear spin $1/2 \Rightarrow$ system of spin- $1/2$ fermions (like electrons in metals)



Liquid ^3He : since 1950 ($^3\text{H} \rightarrow ^3\text{He} + e + \bar{\nu}$ in reactors, weapons)

Below ~ 100 mK, behaves much like (very pure) textbook normal metal (Fermi gas)

$$(c_V \propto T, \chi \propto \text{const}, \eta, D_s \propto T^{-2}, \kappa \propto T^{-1})$$

viscosity

spin diffusion

thermal cond.

but interaction effects v. strong (e.g. $\chi \sim 20$ times value for F.gas)
Why?

Landau Fermi liquid theory:

TD 2.2

start from free Fermi gas, turn on interactions adiabatically

Assume GS evolves adiabatically (excludes superconductivity, ferromagnetism, crystallization...)

Then GS and low excited states **labeled** by quantities $\delta n(\mathbf{p}\sigma)$:

$$\left. \begin{aligned} \delta n(\mathbf{p}\sigma) = 0 \text{ or } 1, & \quad |\mathbf{p}| > p_F \\ \delta n(\mathbf{p}\sigma) = 0 \text{ or } -1, & \quad |\mathbf{p}| < p_F \end{aligned} \right\} \leftarrow \text{Fermi statistics}$$

Expand E of low-lying excited states in $\delta n(\mathbf{p}\sigma)$:

$$E = E_0 + \sum_{\mathbf{p}\sigma} \epsilon(\mathbf{p}\sigma) \delta n(\mathbf{p}\sigma) + \frac{1}{2} \sum_{\substack{\mathbf{p}\mathbf{p}' \\ \sigma\sigma'}} f(\mathbf{p}\mathbf{p}'\sigma\sigma') \delta n(\mathbf{p}\sigma) \delta n(\mathbf{p}'\sigma)$$

\uparrow
 GSE

Symmetry \Rightarrow

$$\left\{ \begin{aligned} \epsilon(\mathbf{p}\sigma) = \epsilon(|\mathbf{p}|) &\cong \epsilon(p_F) + v_F (|\mathbf{p}| - p_F) && : df \ m^* \equiv p_F/v_F \\ f(\mathbf{p}\mathbf{p}'\sigma\sigma') &\cong \sum_{\ell} (f_{\ell}^S + f_{\ell}^A \sigma \cdot \sigma') P_{\ell}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') && \begin{aligned} &\text{"true" DOS} \rightarrow \\ &df \ f_{\ell} = \left(\frac{dn}{d\epsilon}\right)^{-1} \Omega^{-1} F_{\ell}^S, \text{ etc} \\ &\uparrow \\ &\text{volume} \end{aligned} \end{aligned} \right.$$

Informal summary of Landau FL theory:

particles \rightarrow "quasiparticles", fermions with **effective mass** m^*

subject to **molecular fields**, with coeffs. prop. to F_{ℓ}^S, F_{ℓ}^A ,

generated by **polarizations** of the system (example follows)

For real liquid ^3He .

$$m^*/m \sim 3 - 6, \quad F_0^S \sim 10 - 100, \text{ other } F\text{'s mostly } \sim 1.$$

$$\text{(recap): } \left\{ \begin{aligned} E &= E_o + \sum_{p\sigma} \epsilon(p\sigma) \delta n(p\sigma) + \frac{1}{2} \sum_{\substack{pp' \\ \sigma\sigma'}} f(pp'\sigma\sigma') \delta n(p\sigma) \delta n(p'\sigma') \\ f(pp'\sigma\sigma') &\equiv \left(\frac{dn}{d\epsilon} \right)^{-1} \Omega^{-1} \sum_{\ell} (F_{\ell}^s + F_{\ell}^a \sigma \cdot \sigma') P_{\ell}(\hat{p} \cdot \hat{p}') \end{aligned} \right.$$

Select term corr. F_o^a :

$$E_{(ao)} = \frac{1}{2} (dn/d\epsilon)^{-1} \Omega^{-1} F_o^a \sum_{\substack{pp' \\ \sigma\sigma'}} \sigma \cdot \sigma' \delta n(p\sigma) \delta n(p'\sigma')$$

But: $\sum_{p\sigma} \sigma \delta n(p\sigma) = \mathbf{S} \leftarrow$ total spin

$$E_{(ao)} = \frac{1}{2} \Omega^{-1} (dn/d\epsilon)^{-1} F_o^a \mathbf{S}^2$$

Equivalent to molecular field

$$\chi_{mol} = -(dn/d\epsilon)^{-1} F_o^a \mathbf{S}$$

Application: response to external field $\mathbf{H}_{ext}(\mathbf{k};\omega)$

$$\left\{ \begin{aligned} \mathbf{S}(\mathbf{k}\omega) &= \chi_o^{sp}(\mathbf{k}\omega) \chi_{tot}(\mathbf{k}\omega) \\ \mathbf{H}_{tot}(\mathbf{k}\omega) &\equiv \chi_{ext}(k\omega) + \chi_{mol}(k\omega) \\ \mathbf{H}_{mol}(\mathbf{k}\omega) &= - \left(\frac{dn}{d\epsilon} \right)^{-1} F_o^a \mathbf{S}^2(k\omega) \end{aligned} \right. \quad \begin{array}{l} \text{Spin response of} \\ \text{noninteracting F. gas} \\ \text{with mass } m^* \end{array}$$

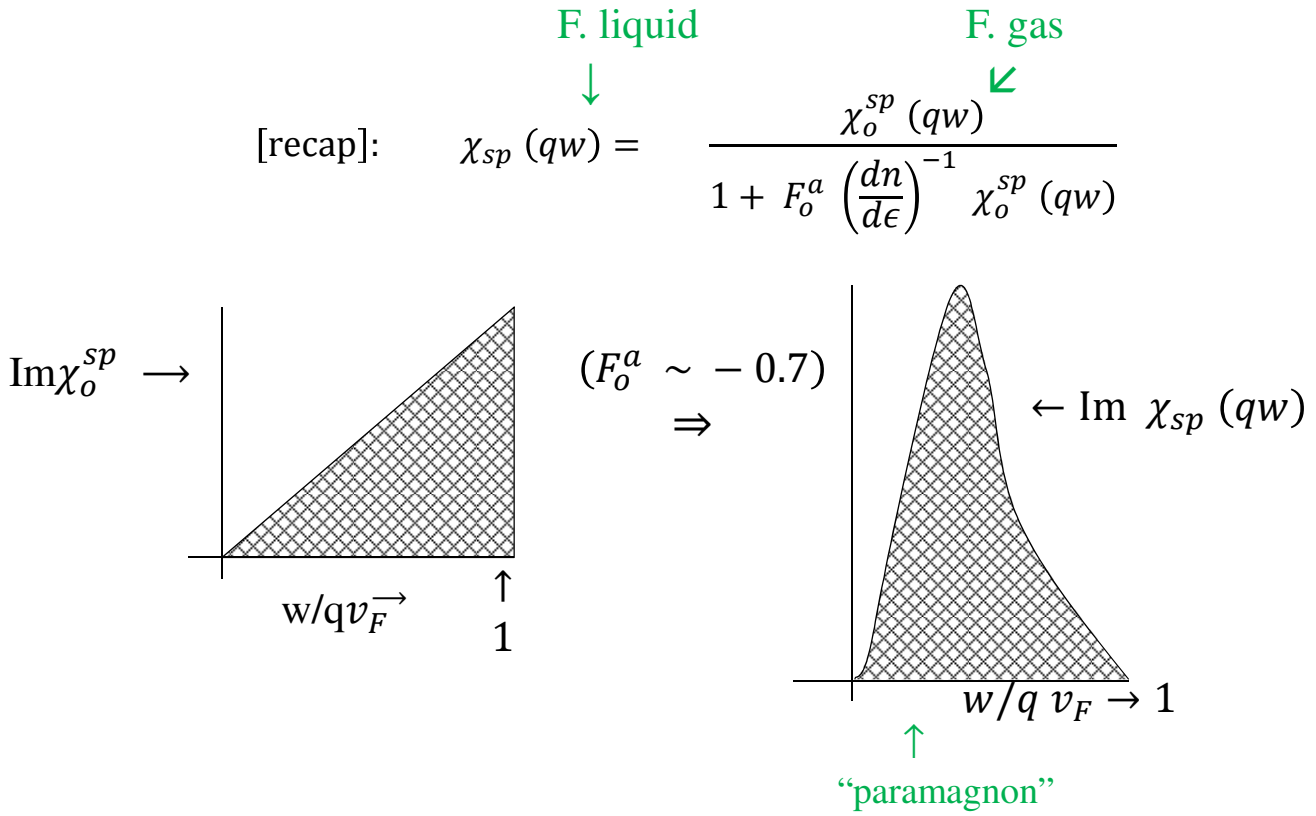
$$\Rightarrow \chi_{true}(q\omega) = \frac{\chi_o^{sp}(q, \omega)}{1 + (dn/d\epsilon)^{-1} F_o^a \chi_o^{sp}(q\omega)}$$

e.g. since static $\chi_o^{sp} = dn/d\epsilon$

$$\chi = \frac{dn/d\epsilon}{1 + F_o^a}$$

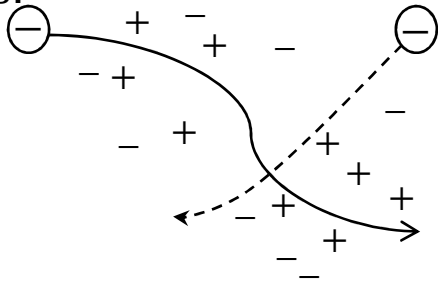
*note spin, field in arbitrary units.

A. Enhanced low-energy spin fluctuations:

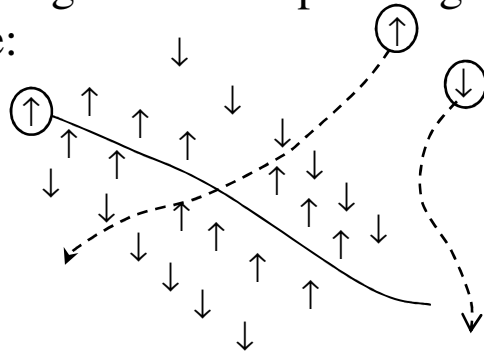


B. Coupling of atomic spins by exchange of virtual paramagnons (V.P's)

Metals:



^3He :



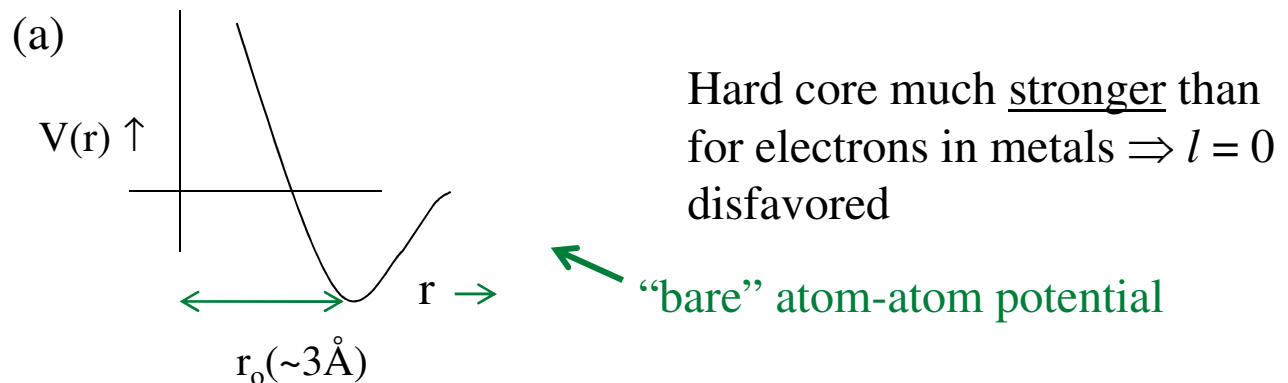
exchange of V.P's induces extra effective spin-spin interaction

$$V_{eff}(\mathbf{q}, w) \sim - (F_o^a)^2 \chi_{sp}(qw) \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}'$$

In limit $q, w \rightarrow 0$ χ_{sp} must be +ve (stability!), hence in this limit

VP-induced interaction always attractive in spin triplet state, repulsive in singlet.

Pairing interaction in liquid ^3He :



Would like to have first max in radial prob. distrn. at $r \sim r_0$.
 But C. pairs formed from states near F. surface $\Rightarrow k \sim k_F$,
 hence first max. at $r \sim l/k_F$.

$$\Rightarrow l \sim k_F r_0 \text{ i.e. } l = 1, 2, \text{ or } 3.$$

↑
orig. favored by theory

(b) Spin-fluctuation exchange \Rightarrow extra term, attractive (repulsive)
 for $S = 1$ ($S = 0$)

(a) + (b) + Pauli principle $\Rightarrow l = 1$ may be favored.

Need to generalize BCS theory to $l \neq 0$ pairing.

1. (for orientation only): anisotropic singlet pairing (e.g. $l = 2$)

BCS ansatz: $\Psi_N = (\sum_k c_k \alpha_{k\uparrow}^+ \alpha_{k\downarrow}^+)^{N/2} |\text{vac}\rangle$, but now

↙ ↘
quasiparticle operators

C_k can depend on **direction** of k on FS.
 otherwise, all the BCS-based maneuvers go through...

Anisotropic singlet pairing (cont)

TD- 2.6

Recap: $\Psi_N = (\sum_k c_k \alpha_{k\uparrow}^+ \alpha_{-k\downarrow}^+)^{N/2} |\text{vac}\rangle,$

c_k function of direction as well as magnitude of k .

All subsequent operations just as in s -wave case, e.g. still have

$$F_k = \Delta_k / 2E_k$$

but F_k (hence $F(\mathbf{r})$) now depends on **direction** of k (\mathbf{r}).

Gap equation: **nontrivially dependent on dirⁿ. of $\mathbf{k}-\mathbf{k}'$.**

$$\Delta_k = - \sum_{k'} V_{kk'} \frac{\Delta_{k'}}{2E_{k'}} \tanh(E_{k'}/2k_B T)$$

Resolve V into spherical harmonic components: for $|k| \sim |k'| \sim k_F$.

$$V_{kk'} = \sum_{\ell} V_{\ell} P_{\ell}(\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')$$

Legendre polynomial

If for some ℓ_0 V_{ℓ} is appreciably larger than for other ℓ 's, then for $|k| \sim k_F$

$$\Delta_k \cong \sum_{\ell m} \Delta_{\ell m} Y_{\ell m}(\hat{\mathbf{k}})$$

spherical harmonic

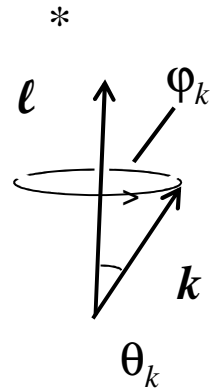
To find coefficients $\Delta_{\ell m}$, minimize (free) energy with respect to possible choices (crudely: for given $|\overline{\Delta^2}|$, minimize $|\overline{\Delta^4}|$)
 $(\bar{A} \equiv (4\Omega)^{-1} \int A(\hat{\mathbf{k}}) d\Omega)$

Note optimal solution may involve nontrivially **complex** $\Delta(\hat{\mathbf{k}})$
 \Rightarrow possible nonzero angular momentum of paired state.

More generally, since for $\ell_0 \neq 0$ $\Delta(\hat{\mathbf{k}})$ must be anisotropic, expect **physical properties anisotropic**

Digression: can an anisotropic Fermi superfluid have (large) angular momentum?

Recap: $\Psi_N = (\sum_k c_k \alpha_{k \uparrow}^+ \alpha_{-k \downarrow}^+)^{N/2} |\text{vac}\rangle \equiv (\Omega^+)^{N/2} |\text{vac}\rangle$
 ↑
 may be complex



Consider e.g. case

$$c_k = f(|k|, \theta_k) \exp 2i\varphi_k \text{ (d-wave)}$$

Naïve approach (ignore boundaries, etc.):

$\hat{L}_z =$ generator of rotations around z-(ℓ -) axis [or explicit calcⁿ.]

$$\Downarrow$$

$$[\hat{L}_z, \hat{\Omega}] = -i\hbar \frac{\partial c_k}{\partial \varphi_k} a_{k \uparrow}^+ \cdot a_{-k \downarrow}^+ = 2\hbar \hat{\Omega}$$

$$\Rightarrow \hat{L}_z \Psi_N = N\hbar \Psi_N$$

Generalization to nonzero T:

anisotropic F. superfluid has **macroscopic** ($\sim N$) ang. momentum at **any** temperature below T_c !

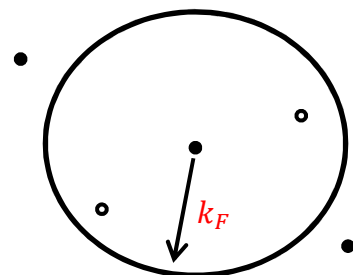
Is this physically correct?

Alternative ansatz:

start with N ground state (F. sea), excite pairs from $|k| < k_F$ to $|k| > k_F$
 (details: AJL QL appendix 6A)

This state has

$$L_z \sim N\hbar(\Delta / E_F)^2 \ll N\hbar$$



Which is right? (expt.?)

Spin triplet pairing

Simplest case (“ESP) ← equal spin pairing

With suitable choice of spin axes, only parallel spins paired ⇒

$$\Psi_N = \sum_k \left(c_{k\uparrow} a_{\mathbf{k}\uparrow}^+ a_{-\mathbf{k}\uparrow}^+ + c_{k\downarrow} a_{\mathbf{k}\downarrow}^+ a_{-\mathbf{k}\downarrow}^+ \right)^{N/2} |\text{vac}\rangle \quad (3)$$

Pauli principle ⇒ $c_{\mathbf{k}\sigma} = -c_{-\mathbf{k}\sigma}$, but in general no particular wherever $c_{\mathbf{k}\uparrow}$ and $c_{\mathbf{k}\downarrow}$.

Note (?) is **coherent** superposⁿ. of $\uparrow\uparrow$ and $\downarrow\downarrow$ pairs, not equivalent to

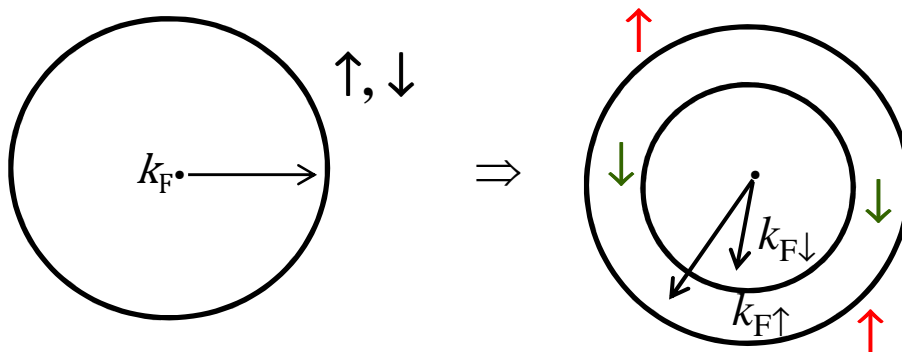
$$\Psi_N^{(F)} = \left(\sum_k c_{k\uparrow} a_{\mathbf{k}\uparrow}^+ a_{-\mathbf{k}\uparrow}^+ \right)^{N/4} \left(\sum_k c_{k\downarrow} a_{\mathbf{k}\downarrow}^+ a_{-\mathbf{k}\downarrow}^+ \right)^{N/4} |\text{vac}\rangle$$

↑ ↑
“Fock” state

Analysis of energetics, etc. parallel to that for singlet pairing, but for spin-conserving potential, $\uparrow\uparrow$ and $\downarrow\downarrow$ gap equations decouple:

$$\Delta_{k\sigma} = -\sum V_{kk'} \Delta_{k'\sigma} / 2E_{k'\sigma} \quad \tanh \frac{1}{2} \beta \epsilon_{k'\sigma}$$

Spin susceptibility of ESP state: in small field \mathcal{H} .



Field does not interfere with pair formation! $\chi_{\text{ESP}} \cong \chi_n$

← N-state χ

Spin triplet pairing: the general case

Most general “BCS-like” triplet-paired state:

$$\Psi_N = \left(\sum_{\mathbf{k}\alpha\beta} c_{\mathbf{k}\alpha\beta} a_{\mathbf{k}\alpha}^+ a_{-\mathbf{k}\beta}^+ \right)^{N/2} |\text{vac}\rangle$$

$$c_{\mathbf{k}\alpha\beta} = c_{\mathbf{k}\beta\alpha} = -c_{-\mathbf{k}\alpha\beta}, \text{ otherwise arbitrary}$$

Since for given \mathbf{k} $c_{\mathbf{k}\alpha\beta}$ is Hermitian, **for given \mathbf{k}** then always exists a choice of spin axes which makes diagonal ($c_{\mathbf{k}\uparrow\downarrow} = c_{\mathbf{k}\downarrow\uparrow} = 0$)

 not necessarily unique

In this set of axes can then define e.g.

$$F_{\mathbf{k}\sigma} \equiv \frac{c_{\mathbf{k}\sigma}}{1+|c_{\mathbf{k}\sigma}|^2}, \text{ etc.}$$

But in general gap eqn. etc. is quite messy. Enormous simplification if state is **unitary**, df. by

$$|c_{\mathbf{k}\sigma}|^2 = \text{ind. of } \sigma, \forall \mathbf{k}.$$

Then can write in arbitrary spin axes

$$\left. \begin{aligned} F_{\mathbf{k},\alpha\beta} &\equiv \Delta_{\mathbf{k},\alpha\beta}/2E_{\mathbf{k}}, \\ E_{\mathbf{k}} &\equiv (\epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2)^{1/2} \\ |\Delta_{\mathbf{k}}|^2 &\equiv \left(\sum_{\beta} |\Delta_{\mathbf{k},\alpha\beta}|^2 \right) \end{aligned} \right\} \text{ ind. of } \alpha$$

For spin-independent potential

$$\langle V \rangle = \sum_{\mathbf{k}\mathbf{k}'} \sum_{\alpha\beta} V_{\mathbf{k}\mathbf{k}'} F_{\mathbf{k}\alpha\beta} F_{\mathbf{k}'\beta\alpha}$$

$$\Rightarrow \text{Gap eqn.: } \Delta_{\mathbf{k}\alpha\beta} = -\sum_{\mathbf{k}'\alpha'\beta'} V_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'\alpha'\beta'}}{2E_{\mathbf{k}'}} \uparrow$$

i.e. gap eqns. For different matrix element $\Delta_{\alpha\beta}$ decoupled

The d-vector notation (unitary states)

Consider e.g. the pair w.f. $F(r; \alpha\beta)$ or its FT $F_{k\alpha\beta}$. For any given \mathbf{k} can find choice of spin axes so that

$$F_{\alpha\beta}(\mathbf{k}) = F_{\uparrow\uparrow}(\mathbf{k}) |\uparrow\uparrow\rangle + F_{\downarrow\downarrow}(\mathbf{k}) |\downarrow\downarrow\rangle$$

and from definition of unitary state.

$$|F_{\uparrow\uparrow}(\mathbf{k})| = |F_{\downarrow\downarrow}(\mathbf{k})| \equiv |F_k|$$

For 2 particles of spin $1/2$ state of this form is given by

$$S = 1, \quad \mathbf{S} \cdot \mathbf{d} = 0$$

\mathbf{d} = real vector up to overall phase in xy-plane making $\angle \frac{1}{2} \arg(F_{\uparrow}/F_{\downarrow})$
with y-axis, ↑ with magnitude equal to $|F_k|$

$$i.e. \mathbf{d} \times \mathbf{d}^* = 0$$

Generalizing to arbitrary reference frame, any unitary phase described by specifying for each \mathbf{k} $\mathbf{d}(\mathbf{k})$ s.t. $\mathbf{d}(\mathbf{k}) \times \mathbf{d}^*(\mathbf{k}) = 0$

Technically, in an arb. reference frame

$$\mathbf{d}(\mathbf{k}) \equiv -i \sum_{i=1}^3 \sum_{\alpha\beta} (\sigma_2 \sigma_i)_{\beta\alpha} F_{\alpha\beta}(\mathbf{k})$$

In BCS case $\mathbf{d}(\mathbf{k})$ is usually a function only of $\hat{\mathbf{n}} = \mathbf{k}/|\mathbf{k}| \cong \mathbf{k}/k_F$.