Liquid Helium-3 and Its Metallic Cousins

Exotic Pairing and Exotic Excitations

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Helium – the simplest element

(electronically completely inert)

^{3}He – arguably the simplest isotope of helium (does not even form diatomic molecules in free space!) (but does have nuclear spin ½)

And yet…. probably more different uses than any other isotope in periodic table!

e.g. gas phase: lung NMR imaging, particle detectors, …
solid phase: thermal vacancies, Pomeranchuk cooling.
nuclear magnetic phase transition…

liquid phase: this talk!
realized in bulk since ~1950 (68 years)
since 1972, superfluid (46 years)

This talk:

1. brief reminders re Cooper pairing in (classic) superconductors (BCS theory)
2. Cooper pairing in superfluid ^{3}He
3. Some idiosyncrasies of uniform superfluid ^{3}He: superfluid amplification (mixture of old and new)
4. A metallic cousin of superfluid ^{3}He (SRO)
5. Some idiosyncracies of inhomogeneous ^{3}He/SRO

Electrons in Metals (BCS):

Fermions of spin ½, $T_F \sim 10^4 K,\ T_c \sim 10K \Rightarrow T_C / T_F \sim 10^{-3}$

$\Rightarrow$ strongly degenerate at onset of superconductivity

Normal state: in principle described by Landau Fermi-liquid theory, but "Fermi-liquid" effects often small and generally very difficult to see.

BCS: model normal state as weakly interacting gas with weak "fixed" attractive interaction

Superconducting state: Cooper pairs form, i.e. :

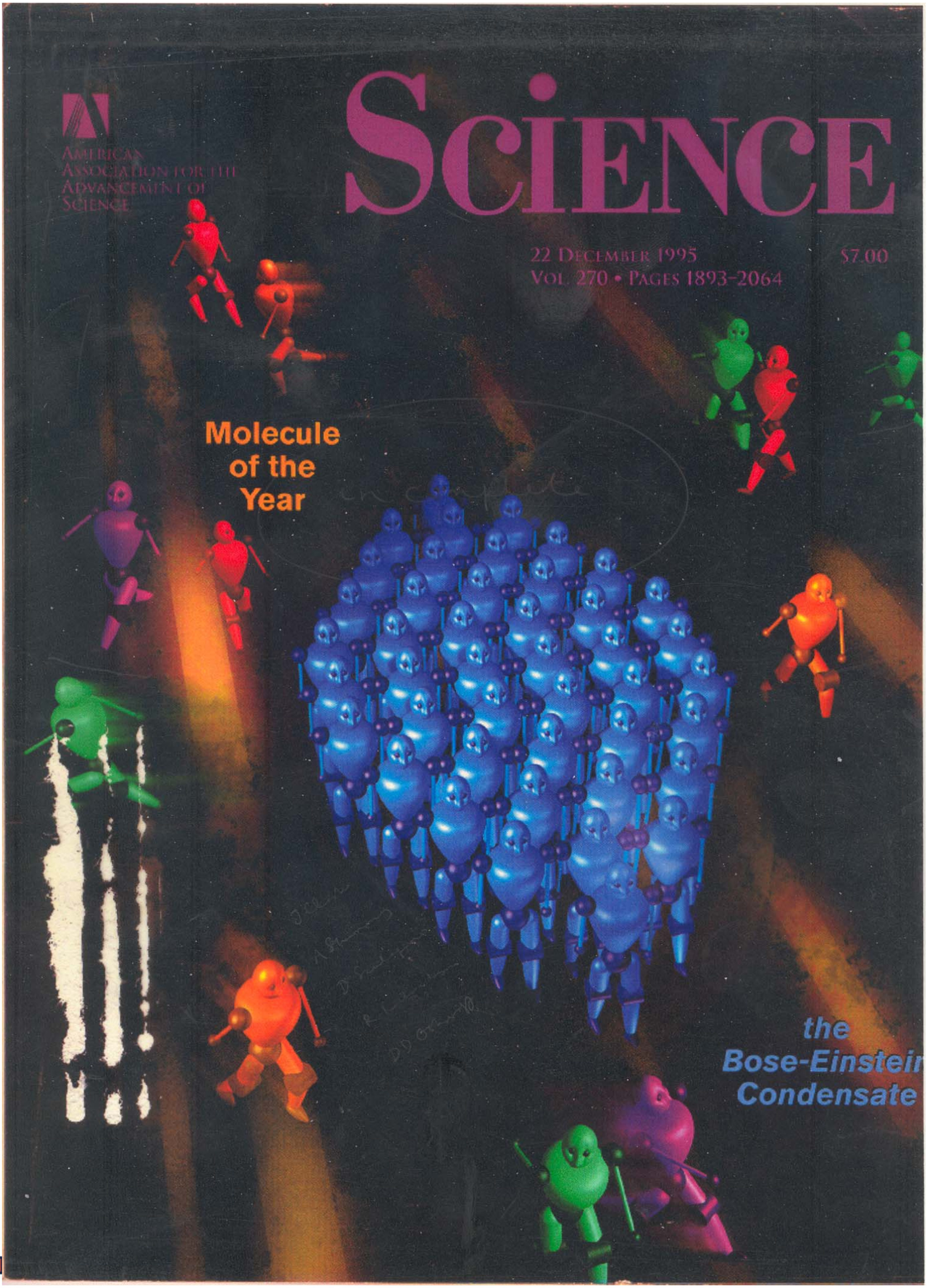
2-particle density matrix has single macroscopic (~N) eigenvalue, with associated eigenfunction

$$\underbrace{F(\boldsymbol{r}_1\boldsymbol{r}_2\sigma_1\sigma_2)}_{\text{“wave function of Cooper pairs”}} \equiv F(\overset{\text{COM}}{\boldsymbol{R}} : \underset{\text{relative}}{\boldsymbol{r}}\sigma_1\sigma_2)$$

$$\left(\equiv \left\langle \psi^\dagger\left(\boldsymbol{R}+\boldsymbol{r}/2:\sigma\right)\psi^\dagger\left(\boldsymbol{R}-\boldsymbol{r}/2:\sigma'\right)\right\rangle\right)$$

in words: a sort of "Bose condensation of diatomic (quasi-) molecules" = a macroscopic number of pairs of atoms are all doing the same thing at the same time ("superfluid amplification")

NW 3



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Molecule of the Year

incomplete

the Bose-Einstein Condensate

D D Osheroff

STRUCTURE OF COOPER-PAIR WAVE FUNCTION

(in original BCS theory of superconductivity, for fixed **R**, σ_1, σ_2)

relative coordinate

$$F(\boldsymbol{r}) = F(|\boldsymbol{r}|) = \Delta\Omega^{-1/2}\sum_{\boldsymbol{k}}\left(2\underbrace{E_k}_{\left(\varepsilon_k^2+|\Delta|^2\right)^{1/2}}\right)^{-1}\ \exp i\boldsymbol{k}\cdot\boldsymbol{r}$$

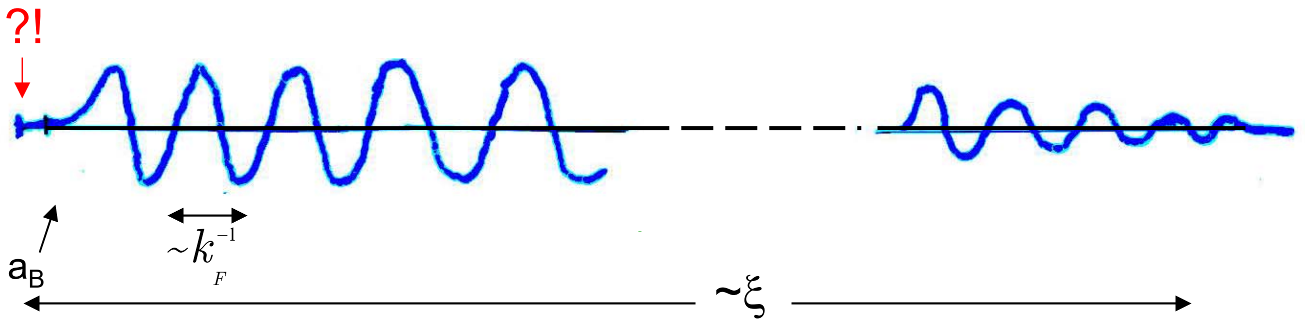
vol.

Energy gap

KE relative to Fermi surface

$$\cong \text{const.}\left(N\Delta/E_F\Omega^{1/2}\right)\frac{\sin k_F r}{k_F r}\ \exp -r/\xi$$

$$\xi = \text{"pair radius"} \sim \hbar\upsilon_F/\Delta\left(\sim 10^4\dot{A}\right)$$



"Number of Cooper pairs" (N_o) = normalization of $F(\boldsymbol{r})$

$$\equiv \int |F(\boldsymbol{r})|^2 \, d\boldsymbol{r} \sim \frac{N^2}{\Omega}\frac{\Delta^2}{E_F^2}\frac{1}{k_F^2}\xi \sim N\left(\Delta/E_F\right) \sim 10^{-4} N$$

(contrast: $N_0/N \sim 10\%$ in ^{4}He)

In original BCS theory of superconductivity,

$$F(\boldsymbol{r}:\sigma_1\sigma_2) = \frac{1}{\sqrt{2}}\left(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2\right)F(|\boldsymbol{r}|)$$

spin singlet

orbital s-wave

⇒ PAIRS HAVE NO "ORIENTATIONAL" DEGREES OF FREEDOM

(⇒stability of supercurrents, etc.)

THE FIRST ANISOTROPIC COOPER-PAIRED SYSTEM: SUPERFLUID ^{3}HE

also fermions of spin ½ $T_F \sim 1K,\ T_c \sim 10^{-3}K \Rightarrow T_C / T_F \sim 10^{-3}$

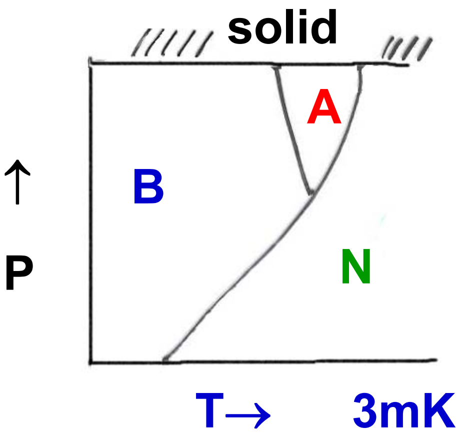
⇒ again, strongly degenerate at onset of superfluidity, but also strongly interacting.

⇒ low-lying states (inc. effects of pairing) must be described in terms of Landau quasiparticles. (and Fermi-liquid effects v. impt.)

2-PARTICLE DENSITY MATRIX ρ_2 still has one and only one macroscopic (~N) eigenvalue

⇒ can still define "pair wave function" $F(\boldsymbol{R},\boldsymbol{r}:\sigma_1\sigma_2)$

However, even when $F \neq F(\boldsymbol{R})$,



$F(\boldsymbol{r}\sigma_2\sigma_2)$ HAS ORIENTATIONAL DEGREES OF FREEDOM!

(i.e. depends nontrivially on $\hat{\boldsymbol{r}},\sigma_1\sigma_2$.)

Standard identifications (from spin susceptibility, ultrasound absorption, NMR… plus theory):

In both A and B phases, Cooper pairs have $\ell = S = 1$

A phase (“ABM”)

Spin triplet

$$F(\boldsymbol{r}:\sigma_1\sigma_2) = \frac{1}{\sqrt{2}}\left(\uparrow_1\downarrow_x + \downarrow_1\uparrow_2\right)_{\hat{\boldsymbol{d}}} \times f(\boldsymbol{r})$$

char. “spin axis”

or with different choice of axes.

$$F\left(\boldsymbol{r}:\sigma_1\sigma_2\right) = \frac{1}{\sqrt{2}}\left(\uparrow_1\uparrow_x + e^{i\chi}\downarrow_1\downarrow_2\right) \times f\left(\boldsymbol{r}\right)$$

violates both P and T!

$$f(\boldsymbol{r}) = f_o(|\boldsymbol{r}|)\times\left(\sin\,\theta\cdot\exp\,i\varphi\right)_{\hat{\ell}} \longleftarrow$$

char. “orbital axis”

apparent angular momentum $\hbar$/pair

Properties anisotropic in orbital and spin space separately,

e.g. $\left|\Delta_K\right| = \left|\Delta\left(\hat{k}\right)\right| = \Delta_o\left|\hat{\boldsymbol{k}}\times\hat{\boldsymbol{\ell}}\right| \Leftarrow$ nodes at $\pm\hat{\ell}$!

B phase ("BW")

For any particular direction $\hat{\boldsymbol{n}}$ (in real or **k**-space) can always choose spin axis s.t.

$$F(\hat{\boldsymbol{n}}:\sigma_1\sigma_2)\sim\frac{1}{\sqrt{2}}\left(\uparrow_1\downarrow_2+\downarrow_1\uparrow_2\right)_{\hat{\boldsymbol{d}}}$$

i.e. $\hat{\boldsymbol{d}}=\hat{\boldsymbol{d}}(\hat{\boldsymbol{n}})$:

Original "theoretical" state had $\hat{\boldsymbol{d}}(\hat{\boldsymbol{n}})=\hat{\boldsymbol{n}}$, i.e. spin of every pair opposite to orbital angular momentum (^3P_o state).

Real-life B phase is ^3P_o state "spin-orbit rotated" by 104°.

L=S=J=O

because of dipole force

cos-1(-1/4)=θ_o

Note: rotation (around axis $\hat{\omega}$) breaks P but not T

$\|$extℓ field $\boldsymbol{H}_o$

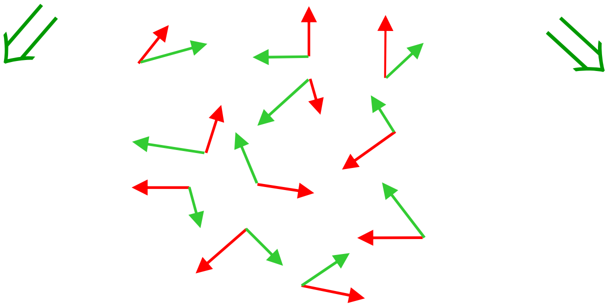
inversion

time reversal

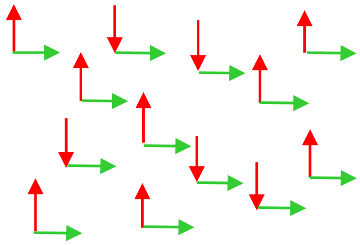
Orbital and spin behavior individually isotropic, but: properties involving spin-orbit correlations anisotropic!

SPIN-ORBIT : ORDERING MAY BE SUBTLE

NORMAL PHASE



A

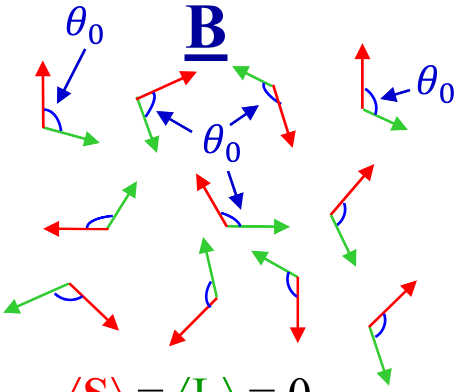


$\langle \mathbf{S}^2 \rangle \neq 0$ (but $\langle \mathbf{S} \rangle = 0$),
$\langle \mathbf{L} \rangle \neq 0$ ($\boldsymbol{d} \,||\, \boldsymbol{\ell}$)

⇐ ORDERED PHASE ⇒

↗ = total spin of pair

↗ = relative orbital ang. momentum)



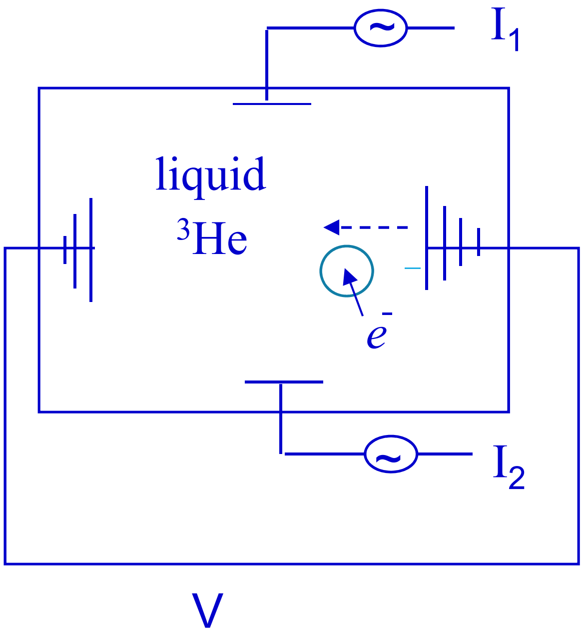
$\langle \mathrm{S} \rangle = \langle \mathrm{L} \rangle = 0$
but $\langle \mathrm{L} \times \mathrm{S} \rangle \neq 0$!
out of screen

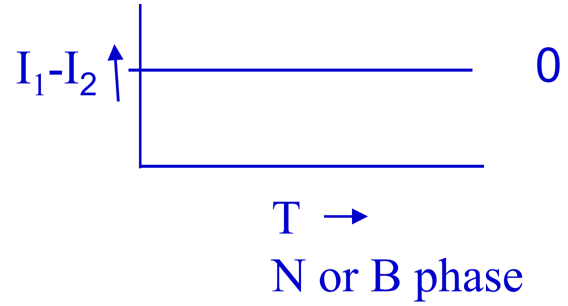
Dipole energy depends on relative angle of ↑ and ↑ ⇒ determines $\hat{d}\cdot\hat{\ell}$ (A phase) or θ_o (B phase)

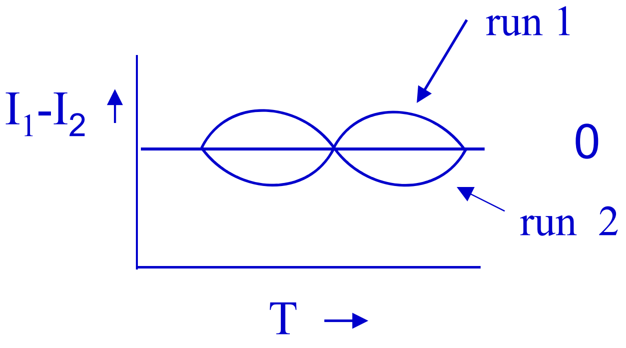
How to "see" the exotic nature of the pairing?

Example*: Spontaneous violation of P- and T-symmetry in A phase

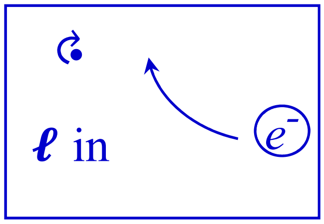
$$\left(f(r) = (\sin\theta e^{i\varphi})_{\hat{\ell}} \right)$$



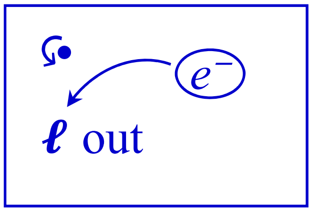




Intrinsic Magnus force:



(run 1)



(run 2)

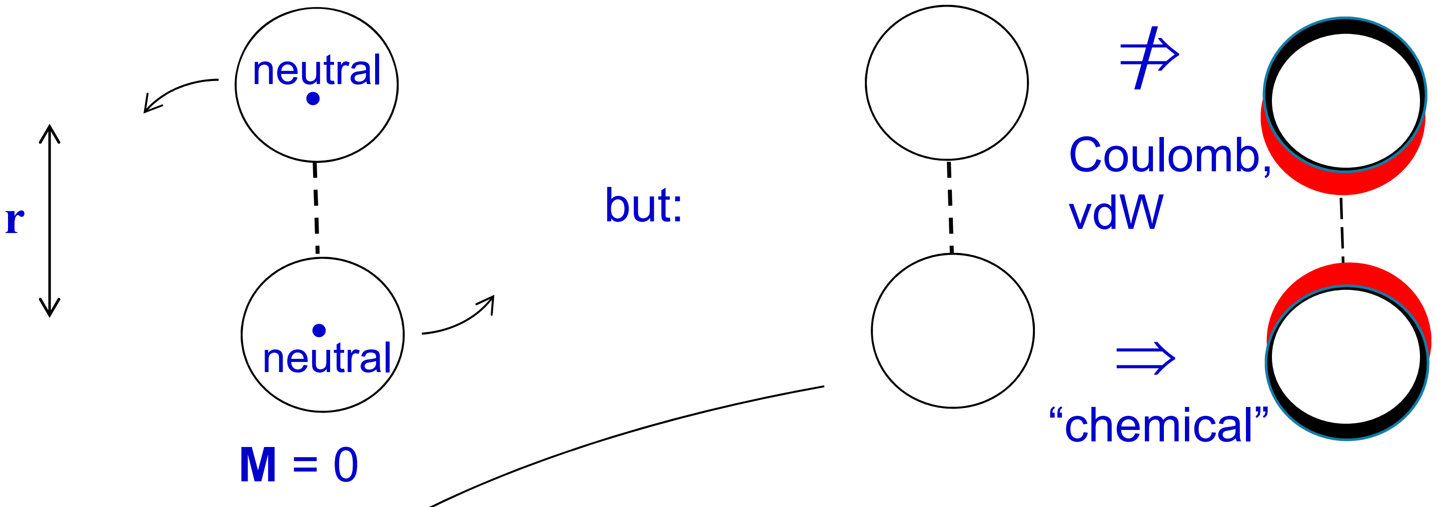
(Somewhat) unexpected effect: magnetic field can orient $\boldsymbol{\ell}$ – vector "in" or "out"!
indicates coupling of $\boldsymbol{\ell}$ to field, i.e. ^{3}He is a weak orbital ferromagnet, with magnetic moment along (±) $\boldsymbol{\ell}$.

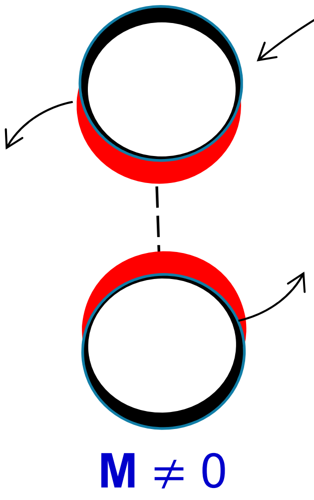
But…. ^{3}He atoms are neutral! How can this be?

*H. Ikegami et al., Science **341**, 59 (2013)

Weak ferromagnetism in ^{3}He – A*

Known effect in chemical physics†: rotation even of homonuclear diatomic molecule gives rise to magnetic moment!





Even in covalent homonuclear diatomic molecules, (e.g. $^2C_{12}$) very tiny effect, moreover falls off exponentially with r:

$$\mu = \mu(r), \mu_{mol} = \int r^2 \underset{\uparrow}{P}(r)\mu(r)dr$$

distribution of radial prob.

In free space, 2 ^{3}He atoms do not even form a bound state! For Cooper pair, vast bulk of $P(r) \equiv |F(r)|^2$ lies at $r \gg a_o, k_F^{-1}$

Hence, for single Cooper pair calculate (lots of exotic chemical physics!) $\mu_{CP} \sim 10^{-11} \mu_B$. (almost certainly immeasurably small). Certainly, in N phase completely unobservable.

What saves us is the principle of superfluid amplification – all Cooper pairs do same thing at same time! As a result, estimate effective equivalent field $H_{eq} = n_{cp}\mu_{CB} / \chi \sim 10-20mG.$. Paulson et al. find circumstantial evidence for spontaneous field of just this o. of m.

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*AJL, Nature **270**, 585 (1977): Paulson & Wheatley, PRL 40, 557 (1978)

†GC Wick, Phys. Rev **73**, 51 (1948)

More spectacular (but less direct) example of superfluid amplification: NMR

Recall: dipole energy depends on angle between ↑ and ↑

dipole energy

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{S} \times \boldsymbol{H}_o + \frac{\delta E_D}{\delta \theta}$$

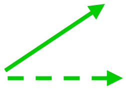
∠of rotation about *rf* field direction $\hat{\mathcal{H}}_{rf}$

$\boldsymbol{H}_o, \hat{\omega}$

↑ $\mathcal{H}_{rf}$ (long) (L)

$\overrightarrow{\mathcal{H}_{rf}}$ (transverse) (T)

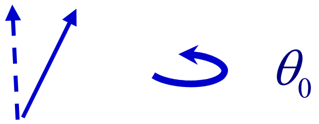
For A phase, dipole energy locks $\boldsymbol{d} \,||\, \ell$ in equilibrium, and usually $\boldsymbol{d} \perp \boldsymbol{H}_o \Rightarrow$ both T and L fields move $\boldsymbol{d}$ away from $\ell \Rightarrow$ T frequency shift + L resonance (√)



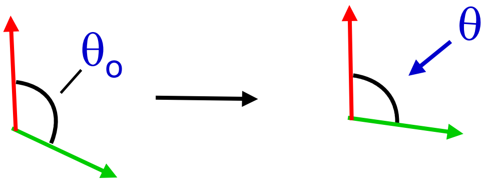
($\boldsymbol{H}_{rf}$ into screen)

For B phase:

in transverse resonance, rotation around $\hat{\mathcal{H}}_{rf}$ equiv. rotation of $\hat{\boldsymbol{\omega}}$ with θ_o unchanged $\Rightarrow$ no dipole torque, $\Rightarrow$ no resonance shift. (√)



In longitudinal resonance, rotation changes θ away from θ_o



⇒ finite-frequency resonance! (√)

One more proposed* (but so far unrealized!) example of superfluid amplification:

P-(but not T-) violating effects of neutral current part of weak interaction:
For single elementary particle, by Wigner-Eckart theorem, any EDM $\boldsymbol{d}$ must be of form

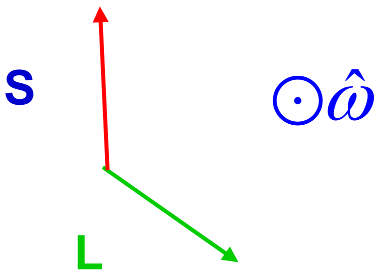
$$\boldsymbol{d} = \text{const.}\ \boldsymbol{J} \qquad \leftarrow \text{violates T as well as P.}$$

But for ^{3}He – B, can form

$$\boldsymbol{d} \sim \text{const.}\ \mathbf{L}\times\mathbf{S} \sim \text{const.}\ \hat{\omega}$$

$$\uparrow$$

violates P but not T.



Calculation involves factors similar to that of A-phase ferromagnetism (lots of even more exotic chemical physics!):

Effect is tiny for single pair, but since all pairs have same value of $\mathbf{L}\times\mathbf{S}$, is multiplied by factor of $\sim 10^{-23} \Rightarrow$

macroscopic P-violating effect?

(maybe in 10-20 years…)

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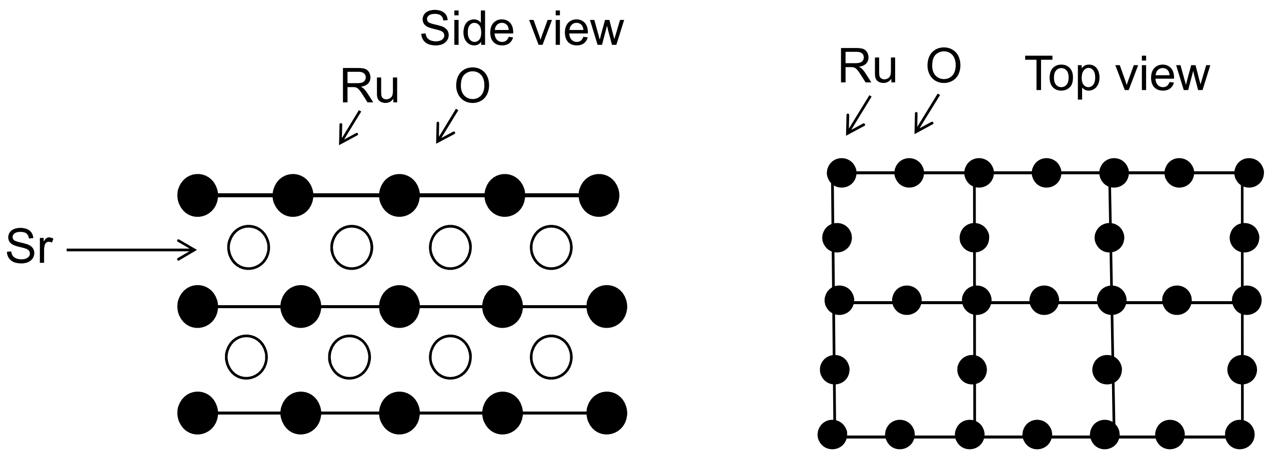
*AJL, PRL **39**, 587 (1977)

A putative metallic cousin of ^{3}He-A: Sr_2RuO_4

(single layer strontium ruthenate, "SRO")

-Strongly layered material, structure similar to cuprates with RuO_2 planes replacing CuO_2.



-Normal state fairly conventional (unlike cuprates)
-Superconducting at $\sim 1 \cdot 5\ K$, strongly type - II.

The $64K question:

What is symmetry of Cooper pairs in S state?

-lots of (partially mutually inconsistent) experimental information (sp. ht., ARPES, μSR, Josephson…), but most plausible conclusion* is

spin triplet, $p_x + ip_y$

If so, then prima facie analogous to A phase of superfluid ^3He, but important differences:

(1) charged system

(2) both $\hat{\boldsymbol{d}}$ and $\hat{\ell}$ vectors can be pinned by lattice.

Nevertheless, some important issues arising in ^{3}He-A have analogs in Sr_2RuO_4 which can be more easily addressed experimentally there. This mostly refers to inhomogeneous phenomena . . .

* C. Kallin, Reps. Prog. Phys. **75**, 042501(2012)

Some (nearly) unique features of spatially inhomogeneous ^{3}He-A / SRO

Recall: pair wave function is spin triplet, so a more general form is

$$f_{\uparrow\uparrow}(\boldsymbol{r})|\uparrow\uparrow\rangle + f_{\downarrow\downarrow}(\boldsymbol{r})|\downarrow\downarrow\rangle$$

Ordinary vortices $f_{\uparrow\uparrow}(\boldsymbol{r}) \sim f_{\downarrow\downarrow}(\boldsymbol{r}) \sim (x+iy)$ well known to occur in both ^{3}He-A and SRO (extreme type-II)

But can also contemplate half-quantum vortex

$f_{\uparrow\uparrow}(\boldsymbol{r}) \sim (x+iy), f_{\downarrow\downarrow}(\boldsymbol{r}) =$ const., i.e. vortex in ↑ spins, none in ↓)

HQV's $f_{\uparrow\uparrow}$ should be stable in ^{3}He-A under appropriate conditions (e.g. annular geom., rotation at $\omega \sim \omega_c / 2, \quad \omega_c \equiv \hbar / 2mR^2$) sought but not found: (in bulk: some recent evidence for ^{3}He in aerogel)

Ideally, would like 2D superconductor with pairing in triplet state. Does such exist? Well, hopefully SRO…

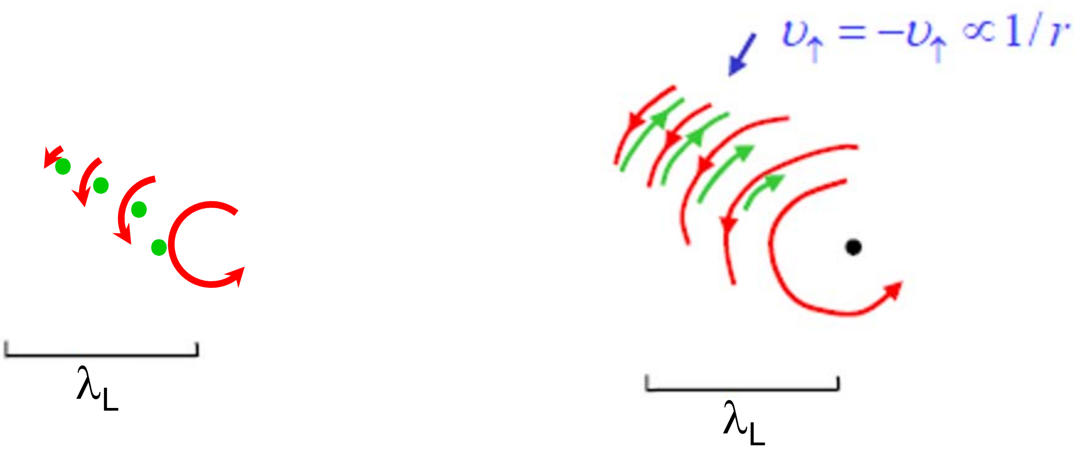
does not need to be p+ip)

Can we generate HQV's in Sr_2RuO_4?

Problem:

in neutral system, both ordinary and HQ vortices have 1/r flow at ∞ ⇒HQV's not specially disadvantaged. But in charged system (metallic superconductor), ordinary vortices have flow completely screened out for $r \gtrsim \lambda_L$ by Meissner effect. For HQV's, this is not true:

London penetration depth



So HQV's intrinsically disadvantaged in Sr_2RuO_4.

Nevertheless Jang et al. (Budakian group, UIUC 2012) find strong evidence for **single** HQV's!

Why not found in ^{3}He-A?

More unique features of inhomogeneous ^{3}He-A and SRO:
(2) ang. momentum and surface currents.

Recall: almost all experimental properties of a degenerate Fermi system, either N or S, are determined by the states near the Fermi surface. In particular, in the S state they are determined by the form of the Cooper pair wave function

$$F(\boldsymbol{R}:\boldsymbol{r},\sigma,\sigma')\left(\equiv\left\langle\psi^{+}\left(\boldsymbol{R}+\frac{\boldsymbol{r}}{2},\sigma\right)\psi^{+}(R-\boldsymbol{r}/2,\sigma'\right\rangle\right)$$

For the S state of both ^3He-A and SRO, the form of F which seems to give best agreement with experiment for homogeneous case $\left(F\neq F(R)\right)$ is

$$F(\boldsymbol{r}:\sigma\sigma')=\left(|\uparrow\uparrow\rangle+e^{ix}|\downarrow\downarrow\rangle\right)\times f(\boldsymbol{r})$$

$$f(\boldsymbol{r})=(\boldsymbol{x}+i\boldsymbol{y})\tilde{f}(|r|)$$

or in Fourier-transformed form for $p\sim p_F$

$$F_p=const.\left(p_x+ip_y\right)\quad\longleftarrow\quad p+ip$$

This appears prima facie to correspond to an angular momentum of $\hbar$/Cooper pair.

However, to obtain the total angular momentum of the system we need the complete many-body wave function. What is this? For infinite system (unrealistic),

Standard answer: (ignore spin degree of function and normalization)

$$\Psi_N = \left(\sum_k C_k a_k^+ a_{-k}^+ \right)^{N/2} |vac\rangle.$$

$N/2$ pairs in vacuum

$(\equiv \hat{k}_x + i\hat{k}_y)$

$$C_k = \hat{c}(|k|) \times \exp i\varphi_k$$

This corresponds to total a.m. $N\hbar/2$, i.e. angular momentum $\hbar/2$ per atom (states of all k contribute, not just those within $\sim\Delta$ of Fermi energy!)

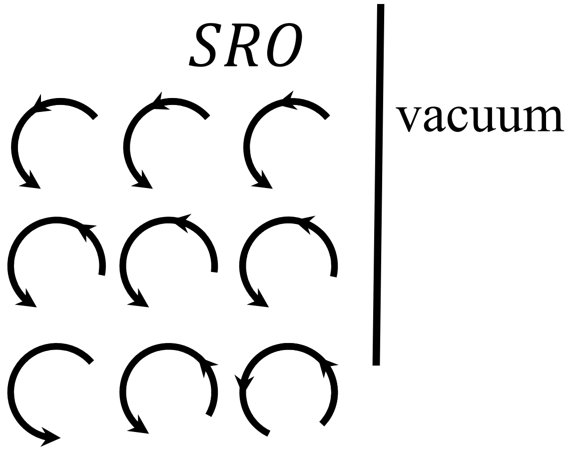
In real life, need to consider system in finite container (e.g. long thin cylinder). What is $\langle \boldsymbol{L} \rangle$?

Theory: 45-year old chestnut! $\big(o(N\hbar),$ $\big(oN\hbar(\Delta/\epsilon_F)\big), o(N\hbar(\Delta/\epsilon_F)^2), 0 \ldots\big)$

Majority opinion is probably $\sim N\hbar$.

What about experiment?

As of now , no direct measurement of $\langle L \rangle$ for either $^3He - A$ or SRO.



However, a somewhat related phenomenon is edge currents: as in ferromagnet, lack of compensation near surface should lead to observable current

For $^3He - A$ this would be a mass current $\Longrightarrow$ difficult to observe. But for SRO, it is an electric current and should produce an observable magnetic field H outside surface.

Matsumoto & Sigrist '99, and many subsequent authors: calculations based on BdG equations give $H\sim$ a few G.

↖ Bogoliubov-de Gennes

Experiments (several groups): upper limit ~1 mG! $\Longrightarrow$ serious problem for "standard" description.

One possible approach: The Cooper-pair wave function may not uniquely determine the many-body wave function!

e.g. to get $F(k) \equiv \langle a_k^+ a_{-k}^+ \rangle \sim \text{const.} \exp i\varphi_k$ for $k \sim k_F$, the "standard" ansatz

$$\Psi_N \sim \left(\sum_k c_k a_k^+ a_{-k}^+ \right)^{N/2} |vac\rangle, \qquad c_k \sim e^{i\varphi_k}$$

may not be unique. Instead of creating ${}^{N}/_{2}$ pairs on vacuum, how about starting from normal Fermi sea and kicking pairs from below Fermi surface to above over range $\sim \Delta \ll \in_F$?

This certainly gives same form of $F(k)$ as standard approach, but gives $\langle L \rangle \sim N\hbar(\Delta/E_F)^2 \ll O(N\hbar)$.

The $64K question: does it yield same BdG equations as standard approach?

More oddities of inhomogeneous ^{3}He-A/SRO:
(3) Majorana fermions

SBU(1)S ↓

In S state, introduce notion of spontaneously broken U(1) symmetry ⇒ particle number not conserved ⇒ (even-parity) GS of form

$$\Psi_{(\text{even})} = \sum_{\substack{N \\ =\text{even}}} C_N \Psi_N$$

⇒ quantities such as $\langle \psi_\alpha(r)\psi_\beta(r)\rangle$ can legitimately be nonzero.

Bogoliubov-de Gennes ↙

Then, mean-field (BdG) Hamiltonian is schematically of form

$$\hat{H}_{mf} =$$

$$\sum_{\alpha\beta}\left\{\underbrace{\int dr\, K_{\alpha\beta}(r)\psi^\dagger_\alpha(r)\psi_\beta(r) + \frac{1}{2}\iint dr\, dr'\Delta_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}')\psi^\dagger_\alpha(r)\psi^\dagger_\beta(r') + HC}_{\text{bilinear in } \psi_\alpha(r),\, \psi^\dagger_\alpha(r):}\right\}$$

(with a term $\mu\delta_{\alpha\beta}$ included in $K_{\alpha\beta}(r)$ to fix average particle number $\langle\hat{N}\rangle$.) $\hat{H}_{mf}$ does not conserve particle number, but does conserve particle number parity, so start from even-parity state.

Interesting problem is to find simplest fermionic (odd-parity) states (“Bogoliubov quasiparticles”). For this purpose write schematically (ignoring (real) spin degree of freedom)

$$\hat{\gamma}_i^{\dagger} = \int \left\{ u_i(r)\hat{\psi}^{\dagger}(r) + v_i(r)\hat{\psi}(r) \right\} dr \qquad \left(\equiv \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} \right) \leftarrow$$

“Nambu spinor”

and determine the coefficients $u_i(r), v_i(r)$ by solving the Bogoliubov-de Gennes equations

$$\left[\hat{H}_{mf}, \hat{\gamma}_i^{\dagger} \right] = E_i \hat{\gamma}_i^{\dagger}$$

so that

$$\hat{H}_{mf} = \sum_i E_i \gamma_i^{\dagger} \gamma_i + \text{const.}$$

(All this is standard textbook stuff…)

Note crucial point: In mean-field treatment, fermionic quasiparticles are quantum superpositions of particle and hole ⇒ do not correspond to definite particle number (justified by appeal to SBU(1)S). This "particle-hole mixing" is sometimes regarded as analogous to the mixing of different bands in an insulator by spin-orbit coupling. (hence, analogy: topological insulator ⇄ "topological superconductor".)

Majoranas – basic element in topological quantum computer?

Recap: fermionic (Bogoliubov) quasiparticles created by operators

$$\gamma_i^\dagger = \int dr\,\{u_i(r)\psi^\dagger(r) + v_i(r)\psi(r)\}$$

with the coefficients $u_i(r), v_i(r)$ given by solution of the BdG equations

$$[\hat{H}_{mf}, \gamma_i^\dagger] = E_i\gamma_i$$

Question: Do there exist solutions of the BdG equations such that

$\gamma_i^\dagger = \gamma_i$ (and thus $E_i = 0$)?

This requires (at least)

1. Spin structure of $u(r), v(r)$ the same ⇒ pairing of parallel spins (spinless or spin triplet, not BCS s-wave)
2. $u(r) = v^*(r)$
3. "interesting" structure of $\Delta_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}')(\sim F(r,r',\sigma,\sigma'))$, e.g. "$p + ip$" $\left(\Delta(\boldsymbol{r},\boldsymbol{r}') \equiv \Delta(\boldsymbol{R},p) \sim \Delta(R)(p_x + ip_y)\right)$

Case of particular interest: “half-quantum vortices” (HQV’s) in ^{3}He-A or Sr_2RuO_4 (widely believed to be $(p + ip)$ superconductor). In this case a M.F. predicted to occur in (say) ↑↑ component, (which sustains vortex), not in ↓↓ (which does not). Note that vortices always come in pairs (or second MF solution exists on boundary). Also, surfaces of ^{3}He-B in certain geometrics.

Why the special interest for topological quantum computing?

(1) Because MF is exactly equal superposition of particle and hole, it should be undetectable by any local probe.

(2) MF’s should behave under braiding as Ising anyons*: if 2 HQV’s, each carrying a M.F., interchanged, phase of MBWF changed by π/2 (note not π as for real fermions!)

So in principle‡:

(1) create pairs of HQV’s with and without MF’s

(2) braid adiabatically

(3) recombine and “measure” result

⇓

(partially) topologically protected quantum computer!

* D. A. Ivanov, PRL **86**, 268 (2001)

‡ Stone & Chung, Phys. Rev. B **73**, 014505 (2006)

Comments on Majorana fermions (within the standard "mean-field" approach)

(1) What is a M.F. anyway?

Recall: it has energy exactly zero, that is its creation operator $\gamma_i^\dagger$ satisfies the equation

$$[H, \gamma_i^\dagger] = 0$$

But this equation has two possible interpretations:

(a) $\gamma_i^\dagger$ creates a fermionic quasiparticle with exactly zero energy (i.e. the odd- and even-number-parity GS's are exactly degenerate)

(b) $\gamma_i^\dagger$ annihilates the (even-parity) groundstate ("pure annihilator")

However, it is easy to show that in neither case do we have $\gamma_i^\dagger = \gamma_i$. To get this we must superpose the cases (a) and (b), i.e.

a Majarana fermion is simply a quantum superposition of a real Bogoliubov quasiparticle and a pure annihilator.

But Majorana solutions always come in pairs ⇒ by superposing two MF's we can make a real zero-energy fermionic quasiparticle

Bog. qp. $\longrightarrow$ $\alpha^\dagger \equiv \gamma_1^\dagger + i\gamma_2^\dagger$

The curious point: the extra fermion is "split" between two regions which may be arbitrarily far apart! (hence, usefulness for TQC)

Thus, e.g. interchange of 2 vortices each carrying an MF ~ rotation of zero-energy fermion by π. (note predicted behavior (phase change of π/2) is "average" of usual symmetric (0) and antisymmetric (π) states)

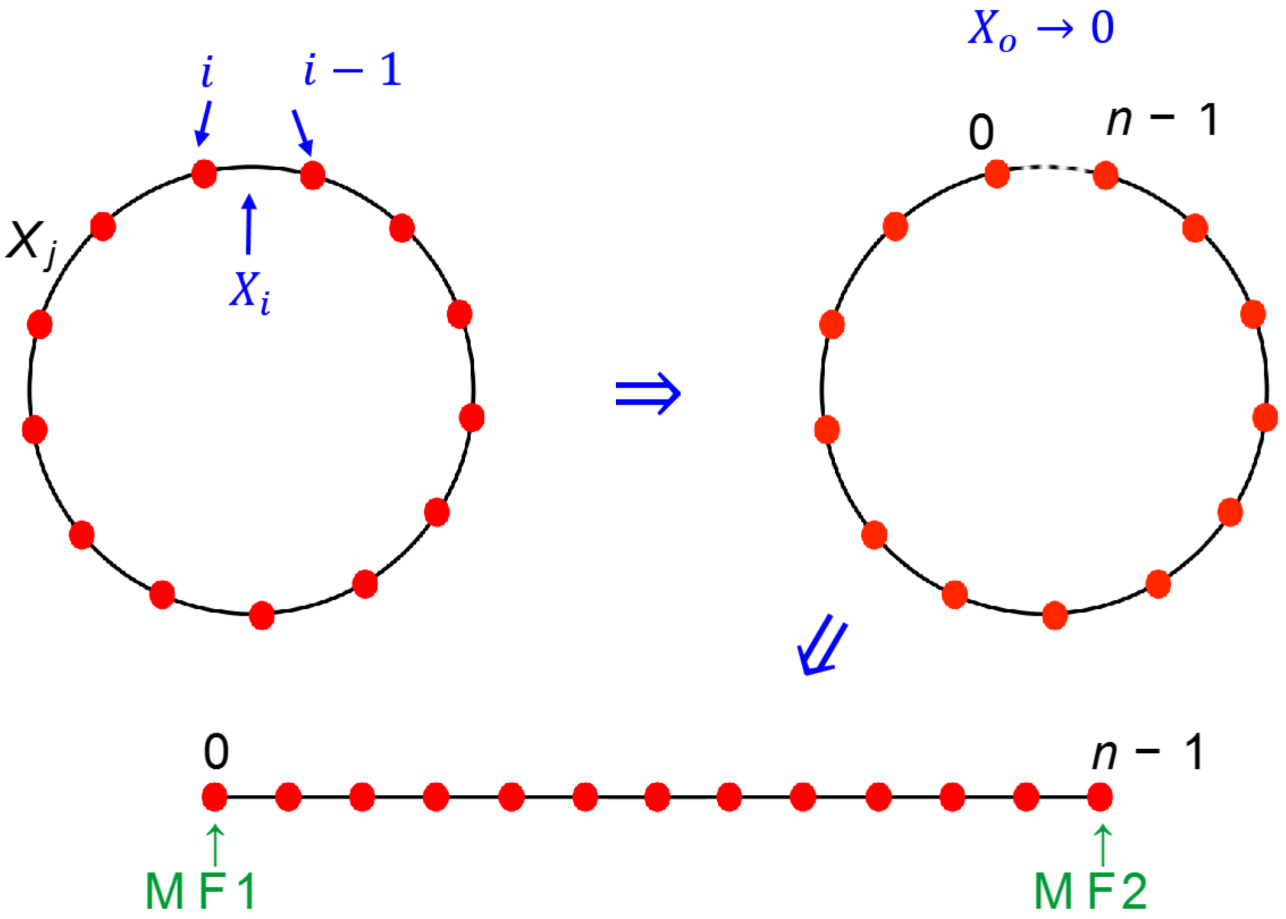
An intuitive way of generating MF’s in the KQW:

Kitaev quantum wire

For this problem, fermionic excitations have form

$$\alpha_i^\dagger = (a_i^\dagger + ia_i) + (a_{i-1}^\dagger + ia_{i-1})$$

so localized on links not sites. Energy for link $(i, i-1)$ is X_i



So far, circumstantial experimental evidence for MF’s in ^{3}He-B: none in ^{3}He-A or SRO. (Rather stronger evidence in artificial systems, e.g. InAs nanowire on Pb.)

Majorana fermions: beyond the mean-field approach

Problem: The whole apparatus of mean-field theory rests fundamentally on the notion of SBU(1)S ← spontaneously broken U(1) gauge symmetry:

$$\Psi_{\text{even}} \sim \sum_{\substack{N= \\ \text{even}}} C_N \Psi_N \qquad \left(C_N \sim |C_N| e^{iN\varphi}\right)$$

$$\Psi^{(c)}_{\text{odd}} \sim \int dr \left\{u(r)\hat{\psi}^\dagger(r) + v(r)\hat{\psi}(r)\right\}|\Psi_{\text{even}}\rangle \quad \left(\equiv \hat{\gamma}_i^\dagger |\Psi_{\text{even}}\rangle\right)^{*}$$

But in real life condensed-matter physics,

SB U(1)S IS A MYTH!!

This doesn't matter for the even-parity GS, because of "Anderson trick":

$$\Psi_{2\text{N}} \sim \int \Psi_{\text{even}}(\varphi) \exp -iN\varphi \, d\varphi$$

But for odd-parity states equation (*) is fatal! Examples:

(1) Galilean invariance

(2) NMR of surface MF in ^{3}He-B

We must replace (*) by

creates extra Cooper pair

$$\hat{\gamma}_i^\dagger = \int dr \left\{ u(r)\hat{\psi}^\dagger(r) + v(r)\hat{\psi}\underset{\uparrow}{C}^\dagger \right\}$$

This doesn’t matter, so long as Cooper pairs have no “interesting” properties (momentum, angular momentum, partial localization...)

But to generate MF’s, pairs must have “interesting” properties!

⇒ doesn’t change arguments about existence of MF’s, but completely changes arguments about their braiding, undetectability etc.

May need completely new approach!