# Shanghai Jiao Tong University
# Lecture 3
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# Bose – Einstein Condensation ("BEC")

Recall: for a system of (structureless, spinless) bosons, wave function must be totally symmetric under interchange of coordinates of only two particles:

$$\Psi(\boldsymbol{r}_1\boldsymbol{r}_2..\boldsymbol{r}_i \dots \boldsymbol{r}_j \dots .\boldsymbol{r}_N) = \Psi(\boldsymbol{r}_1\boldsymbol{r}_2 \dots .\boldsymbol{r}_j \dots .\boldsymbol{r}_i..\boldsymbol{r}_N)$$

For a gas of noninteracting particles, this leads to Bose-Einstein statistics: for particles whose total number is not conserved (*e.g.* photons)

$$n_k = (\exp\beta\mathcal{E}_k - 1)^{-1} \qquad (\beta \equiv 1/k_B T)$$

But if total number is conserved (*e.g*. He atoms)

$$n_k = (\exp\beta\,(\epsilon_k - \mu) - 1)^{-1}$$

chemical potential ⟵ $, \leq 0$ (since $n_k \geqslant 0$)

where $\mu$ is fixed so as to satisfy

$$\sum_k n_k(\mu{:}\,T) = N$$ ⟵ total number of particles

At high $T$, $\mu$ is negative and large. As $T$ falls, since $(\partial N/\partial\mu)_T > 0$ and $(\partial N/\partial T)_\mu > 0$, to keep $N$ constant, $\mu$ must increase. But maximum value of $N$ is obtained, for given $T$, when $\mu = 0$ and is given by

$$N_{max}(T) = \sum_k (\exp(\beta\epsilon_k) - 1)^{-1} \qquad (\propto VT^3)$$

Hence for any given total number and volume, there exists a temperature $T_O$ such that $N_{max}(T_O) = N$. Below this temperature in thermal equilibrium, lowest-energy single-particle state (usually $\boldsymbol{k} = 0$) is macroscopically occupied:

$n_o \sim N$ ← definition of BEC

For a gas of charged (but noninteracting) bosons on a ring in flux $\Phi$, single-particle energies are given by

$$E_\ell = \text{const.}\left(\ell - \Phi/\Phi_o^{sp}\right)^2, \quad j_\ell = \text{const.}\left(\ell - \Phi/\Phi_o\right) \qquad \ell = 0, \pm 1, \pm 2 \ldots$$

So when BEC takes place, it does so in the state which minimizes $E(\ell)$, *i.e.* that with $\ell$ the next integer to $\Phi/\Phi_o^{sp}$; in particular, for $\Phi < \frac{1}{2}\Phi_o^{sp}$, condensation is into $\ell = 0$ state, and contributes an amount $\propto N$ to the circulating current.

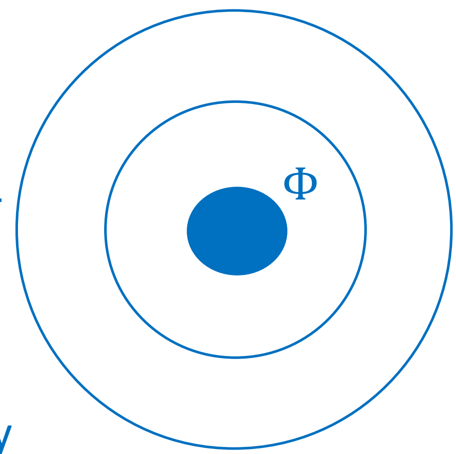
Thus, prima facie, if one could invoke BEC one could explain Meissner effect (also vanishing Peltier effect, since a single state carries no entropy). But…

Two problems with BEC as an explanation of superconductivity:

1. Does not (by itself) explain metastability of supercurrents.
2. Electrons are not bosons but fermions!

1. The problem of supercurrent metastability

A. Formulation of problem:




Imagine we start with the system in equilibrium at $T = 0$ with some trapped flux $\Phi > \frac{1}{2}\Phi_o^{sp}$. According to previous analysis, the single-particle energy levels and currents are given by

$$\left.\begin{aligned} \in(\ell{:}\Phi) &= \frac{\hbar^2}{2mR^2}\left(\ell - \Phi/\Phi_o^{sp}\right)^2 \\ j(\ell;\Phi) &= \frac{e\hbar}{mR}\left(\ell - \Phi/\Phi_o^{sp}\right) \end{aligned}\right\} \quad (\ell = 0, \pm 1, \pm 2 \ldots)$$

so BEC takes place in the state which has $\ell$ closest to $\Phi/\Phi_o^{sp}$; by construction this $\ell \neq 0$. Now we adiabatically turn off the flux: in this process we assume $\ell$ does not change, thus the final value of $\ell$ is still $\neq 0$. But since $\Phi$ is now zero, we now have

$$\in'(\ell) = \frac{\hbar^2}{2mR^2}\ell^2 \qquad \left(\text{and } j'(\ell) = \frac{e\hbar}{mR}\,\ell\right)$$

and it is clear that the GS has $\ell = 0$, so that our resultant state $(\ell \neq 0)$ cannot be stable.

B. Can the system relax to $\ell = 0$? Prima facie, situation is exactly similar to relaxation of excited state of single electron in atom.* ($\tau \sim 10^{-9}$ secs!) In that case, looking at behavior of wave function in xy-plane, for $\ell = 1$ ($p -$ state).

$$\left.\begin{array}{r}\psi_{in}(\theta) \equiv \psi_p(\theta) \sim \exp i\theta \\ \psi_f(\theta) \equiv \psi_s(\theta) \sim \text{const.}\end{array}\right\} \text{“topologically” distinct}$$

Let’s form a function which interpolates between these forms:

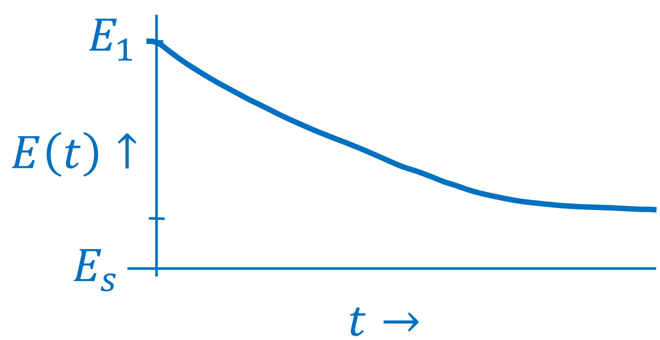
$$\psi\,(\theta\!:t) = a(t)\psi_p(\theta) + b(t)\psi_s(\theta)$$

$$\begin{cases} |a(t)|^2 + |b(t)|^2 = 1 \\ a(-\infty) = 1, b(-\infty) = 0 \\ a(+\infty) = 0, b(+\infty) = 1 \end{cases}$$

Because of linearity of Schrödinger equation

$$E(t) = |a(t)|^2 E_\rho + |b(t)|^2 E_s$$

– downhill all the way! Note always ∃ a value of $\theta$ and $t$ for which we get a node. For the free Bose gas, can do exactly the same for the single-particle state $\psi(\theta\!:t)$ in which BEC is realized



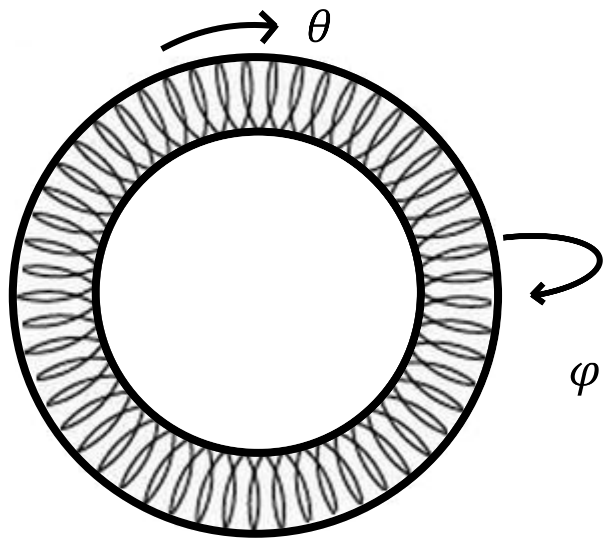

$\Rightarrow$ no metastability.

Yet in experiment, $\tau \gtrsim 10^{15}$ secs!!

*with EM field treated as classical.

# Stability of supercurrents:

## C. Topological argument




Let's consider a more general annular geometry, so that angular momentum is not necessarily conserved. Nevertheless, for any single-particle wave function $\psi(\theta)$ (with $\psi(\theta + 2\pi) = \psi(\theta)$) we can define

(for any point at which $|\psi(\theta)| \neq 0$)

$$\varphi(\theta) \equiv \arg \psi(\theta)$$

and thus the winding number

$$\mathcal{N} \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \varphi(\theta)}{\partial \theta} d\theta \equiv \frac{\phi(2\pi) - \varphi(0)}{2\pi} = 0, \pm 1, \pm 2 \ldots$$

(because $\phi(2\pi) - \varphi(o)$ mod. $2\pi$)

(Analogy: string wound around hula-hoop)

Crucial point: $\mathcal{N}$ is topologically conserved, *i.e.* the only way to change it is to "cut the string" (*i.e.* let $|\psi(\theta)| \rightarrow 0$ for some value of $\theta$). This is exactly what the electron in the atom did… Why can the condensate not do the same?

In Schrödinger mechanics energy is bilinear in $\psi, \psi^*$. What if we add a quartic term, so that $E\{\psi\} = E_{schr} + \int_o^{2\pi} \kappa\, |\psi\,(\theta)|^4 d\theta$? Then for $p \to s$ transitions repeat interpolation

$$\psi(\theta{:}\, t) = a(t)\psi_p(\theta) + b(t)\psi_s(\theta), \qquad |a(t)|^2 + |b(t)|^2 = 1.$$

Now we have:

$$E(b) = E_{Schr}(t) + \int_o^{2\pi} d\theta\kappa\, |\psi\,(\theta{:}\, t)|^4$$

$$= E_{Schr}\,(t) + \kappa \int_o^{2\pi} \left|a(t)e^{i\theta} + b(t)\right|^4 d\theta/2\pi$$

The term in κ *is*

$$\kappa \int_0^{2\pi} d\theta/2\pi \left\{[a]^2 + [b]^2 + 2Re\left(ab^* e^{i\theta}\right)\right\}^2$$

$$\equiv \kappa \int \frac{d\theta}{2\pi} \left(1 + 2|a| \cdot |b| cos(\theta - \theta_0)\right)^2 \qquad \theta_0 \equiv -\arg ab^*$$

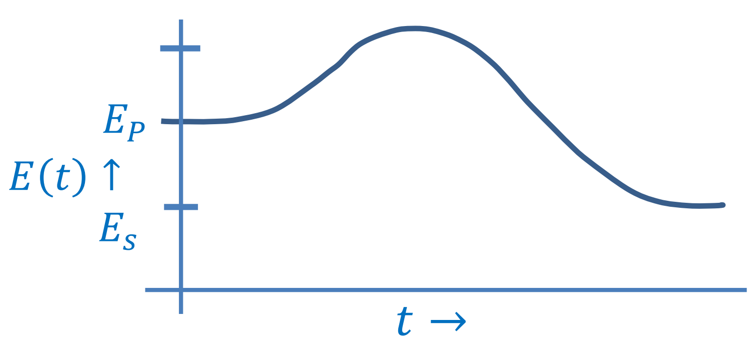
$$= \kappa(1 + 2|a|^2 \cdot |b|^2)$$

Hence

$$E(t) = \kappa + E_{Schr}(t) + 2\kappa|a(t)|^2 \cdot |b(t)|^2$$

$$\equiv const. = -(|b(t)|^2 - |a(t)|^2)\left(E_p - E_s\right) + 2\kappa|a(t)|^2 \cdot |b(t)|^2$$

which for $\kappa > \left(E_p - E_s\right)$ is nonmonotonic ⇒ metastability!



## 2. The problem of statistics: the "BEC-BCS crossover"

Recap: 2 fermions in free space, interacting via short-range attractive potential $v(r)$ of controllable strength with range $\sim r_o$. If fermions, form spin singlet$\left(\frac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)\right)$, then spatial wave function symmetric → s-state possible.
Parametrize by scattering length $a_s$.

1. For sufficiently strong attraction, particles form bound state ("molecule") with radius $\sim r_o$, binding energy $\gtrsim \hbar^2/mr_o^2$. $a_s$ is small and positive. (2 fermions = 1 boson)
2. As attraction weakened, radius becomes $\gg r_o$ and eventually $\rightarrow \infty$. At this point also $a_s \rightarrow +\infty$.
3. Beyond this point $a_s$ is negative and no bound state is formed.

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Expectations in system of N such fermions (assume $nr_o^3 \ll 1$ *i.e.* "dilute") *e.g.* ultracold Fermi alkali gas

1. In region where $a_s$ is positive and small, since $r_o \ll$ interparticle spacing, can treat as bosons and (at $T = 0$) expect simple BEC of molecules. – no effect of Fermi statistics.
2. As $a_s \rightarrow +\infty$, "molecules" start to overlap strongly, so expect nontrivial effects of (a) inter-molecular interactions and (b) underlying Fermi statistics. But plausible that (some kind of) BEC possible.
3. The $64K question: What happens for $a_s \rightarrow \infty$ ("unitarity")?

Apparent answer (from theory and experiment in ultracold Fermi gases)

nothing! *i.e.* in many-particle system, onset of 2-particle bound state is just not seen.

in fact, now believed that "BEC of pairs" persists right up to "BCS limit" ($a_s$ negative and small), *i.e.* ultraweak attraction)

Why?

Partial clue: statements for 2-particle system are valid only in 3D. In 2D or 1D a bound state is formed for arbitrarily weak attraction (but in 2D case, binding energy exponentially small in interaction strength).

So: can we regard superconductivity as a sort of BEC of pairs of electrons?

## Summary of lecture 3

Bose-Einstein condensation (BEC) can explain Meissner effect, but

(a) cannot by itself explain metastability of supercurrents.

(b) Electrons are fermions not bosons. ("statistics problem")

Metastability of supercurrents can be explained if there is a term in the energy proportional to $|\psi(r)|^4$ with a positive coefficient. "Statistics problem" might be explained if tightly-bound difermionic molecules evolve smoothly into much more weakly bound collective state.