Shanghai Jiao Tong University
Lecture 10
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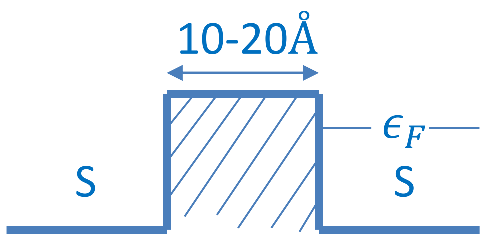
The Josephson Effect

Josephson effect occurs when 2 bulk superconductors connected by weak link, i.e. region which allows passage of electrons but with (much) increased difficulty.

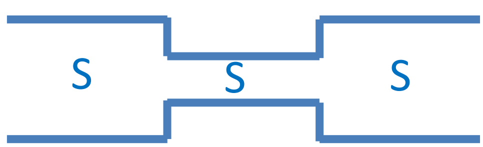
Examples:

(1) Tunnel oxide (S-I-S) junction: schematically,

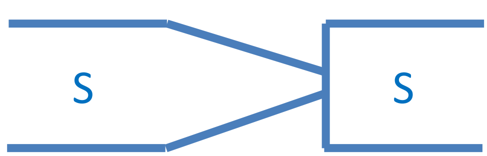


(2) Proximity (S-N-S) junction

(3) Constriction ('microbridge')



(4) Point contact



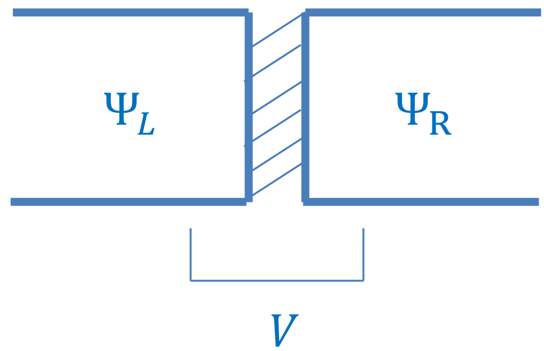
Original Josephson predictions made for case (1), often but not always valid for other cases.

2 equations involving $\Delta\varphi$ (drop in phase of order parameter Ψ across junction, i.e. arg $(\Psi_L^* \Psi_R)$:

1. $I = I_c \sin\Delta\varphi$ ← disspationless supercurrent

↑

critical current, $\sim 1\text{ nA} - 1\text{ mA}$



2. $\frac{d}{dt}\Delta\varphi = \frac{2eV}{\hbar}$ ← voltage (electrochemical potl) drop across junction

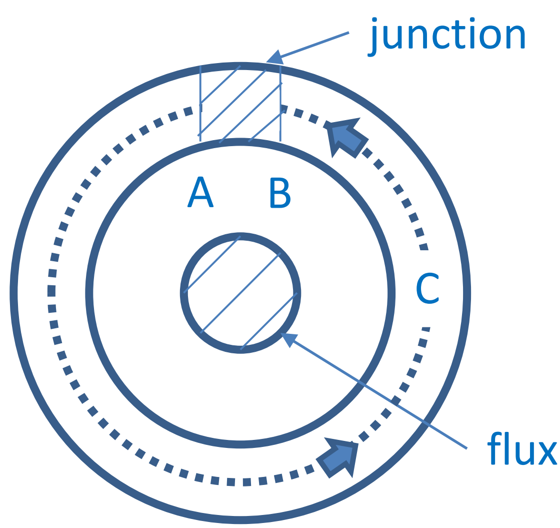
➡ (a) dc Josephson effect (current bias):
$\Delta\varphi = \sin^{-1}(I/I_c)(I \leq I_c)$
(b) ac Josephson effect (voltage bias):
$I = I_c \sin(\frac{2eV}{\hbar})t$

Fundamental significance of Josephson effect: critical current I_c corresponds (see below) to characteristic energy $I_c\Phi_0/2\pi \sim$ 20 mK - 20,000 K, i.e. often $\ll k_B T_{\text{room}}$, yet this tiny energy controls aspects of the system (e.g. trapped flux in ring) which are by any reasonable definition macroscopic! (see below)

"Derivation" of second Josephson equation (Bloch):

Assume thickness of ring $\gg \lambda_L$ but self-inductance negligible (i.e. $LI_c \ll \Phi_0$) so for $I \leq I_c$ flux through ring is externally applied AB flux Φ.

If contour C is well inside penetrations depth, the electric current $J(r)$ on $C = 0$. But $J(r) \propto (\nabla\varphi(r) - 2eA(r)/\hbar)$, so $\nabla\varphi(r) = 2eA(r)/\hbar$. We can integrate from A to B and, since thickness of junction << radius of ring, extend integral of RHS to full circle, whereupon it gives $2\pi\Phi/\Phi_0$. Hence, defining $\Delta\varphi$ as phase drop from A to B :



$$\Delta\varphi = 2\pi\Phi/\Phi_0$$ ⟵ fundamental relation for Josephson circuits

Differentiation with respect to time gives

$$\frac{d}{dt}\Delta\varphi = \frac{2\pi}{\Phi_0}\frac{d\Phi}{dt},$$

voltage drop across junction

but by Faraday's law $d\Phi/dt = V_{\text{circ}}(t) = V(t)$, so

since bulk superconductor shorts out V_{circ}

so $$\frac{d}{dt}\Delta\varphi = \frac{2\pi}{\Phi_0}V \equiv \frac{2eV(t)}{\hbar}$$ i.e. 2nd Josephson equation

This equation is rather generally valid for any kind of weak link.

Derivation of 1st Josephson equation (also Bloch):

Consider dependance of free energy $F(\Phi : T)$ of system on AB flux Φ. By Byers-Yang theorem, F must be periodic in Φ with period $h/e \equiv \Phi^{sp} \equiv 2\Phi_0$. Also, by TRI.

↑ (time-reversal invariance)

$F(\Phi) = F(-\Phi)$. Hence

$F(\Phi, \mathrm{T}) = \sum_n A_n(T) \cos(2\pi n\Phi/2\Phi_0)$ and since $\Delta\varphi = 2\pi\Phi/\Phi_0$

$$F(\Phi : T) = \sum_{n=0}^{\infty} A_n(T) \cos(n\Delta\varphi/2)$$

↑ quite generally valid, independently of nature of weak link.

Suppose t is the matrix element for a single electron (not a Cooper pair!) to traverse the barrier. Then by an extension of the argument used above to obtain $\Delta\varphi = 2n\Phi/\Phi_0$, we see that the term in n involves n single-particle traversals of the ring, and thus of the barrier, so that the amplitude for this process $\propto t^n$. Since the term in $n = 0$ is independent of Φ , the first "interesting" term is apparently n=1.

This term does occur in "mesoscopic" rings; however, it requires phase coherence of the single-electron wave functions around the ring, $\Rightarrow R \lesssim l_\varphi$.

↑ “phase breaking” length

Hence in practice for most systems involving Josephson junctions, the term n=1 (and more generally odd-n) is negligibly small. Then changing the notation so that n/2 -> n, we find

$$F(\Phi, T) = \sum_{n=0}^{\infty} A_n(T)\cos(n\Delta\varphi) \qquad A_n(T) \propto t^{2n}$$

↑

general for weak links with no single-electron phase coherence

Simplest case is $t \to 0$ (typical for tunnel oxide junctions, original Josephson case): then neglect $n > 1$ and find (- sign for convenience!)

$$F(\Phi, T) = -E_{\mathrm{J}}\cos(\Delta\varphi)$$

However, quite generally we have for the current flowing in the ring (and thus through the junction)

$$I = \partial F/\partial\Phi = \frac{2\pi}{\Phi_0}\partial F/\partial(\Delta\varphi)$$

and so

$$I = I_c \sin\Delta\varphi \qquad I_c = (2\mathrm{n}\Phi_0)E_{\mathrm{J}} \equiv (2e/\hbar\,)E_{\mathrm{J}}$$

i.e. the 1st Josephson equation.

In the more general case ($t \nrightarrow 0$) the free energy need not be a single-valued function of $\Delta\varphi$ $(\mathrm{mod}\ 2\pi)$. Suppose by applying constant V we "crank up" $\Delta\varphi$. What happens at $\Delta\varphi = \pi$?

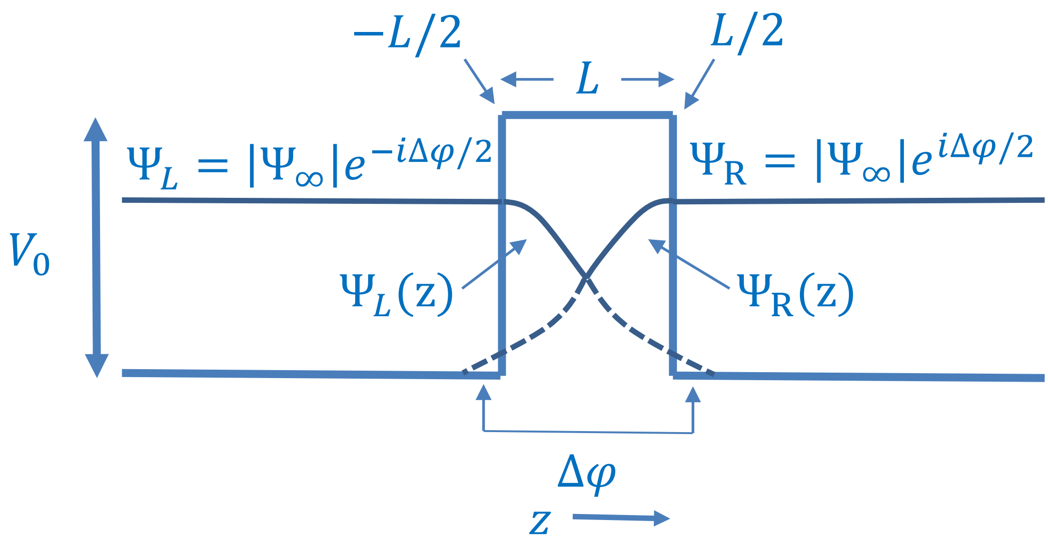
(a) nonhysteretic case (original Josephson case): $|\Psi| \to 0$ in junction ⇒ phase drop $\Delta\varphi$ "slips" from π to $-\pi$

(b) hysteretic case (typical e.g. for microbridge): $|\Psi| \nrightarrow 0$, so $\Delta\varphi$ increases beyond π with increase of KE. Eventually (cf. l. 4) $|\Psi|$ decreases and I reaches a max value I_c

<u>What determines I_c (i.e. E_J)?</u>

(a) "Toy" model of tunnel-oxide junction: extend GL description under barrier ($|z| < L/2$) but since $|\Psi(z)| \to 0$, neglect term in $|\Psi|^4$

➡ Schrodinger problem, with mass $m_p \equiv 2m_{el}$, potential $V_p(z) \equiv 2V_{el}(z)$, energy eigenvalue $\mu_p \equiv 2\mu_{el}$



In limit $L \to \infty$, solution starting from LHS is

$$\Psi_L(z) = \Psi_L(\infty)\exp\left(-\int_{-\frac{L}{2}}^{z} \sqrt{2m_p(2V_0 - 2\mu)}dz\right)$$

$$\Psi_L(\infty) \equiv |\Psi_\infty| e^{-i\Delta\varphi/2}$$

and thus starting from RHS is

$$\Psi_R(z) = \Psi_R(\infty)\exp\left(-\int_{z}^{\frac{L}{2}} \sqrt{2m_p(2V_0 - 2\mu)}dz\right)$$

$$\Psi_R(\infty) \equiv |\Psi_\infty| e^{+i\Delta\varphi/2}$$

For $L \nrightarrow \infty$, must superpose:

$$\Psi(z) = \Psi_L(z) + \Psi_R(z) \equiv \Psi_{\Delta\varphi}(z)$$

The dependence of the energy on $\Delta\varphi$ must come entirely from the cross-terms between Ψ_R and Ψ_L

$$E_J(\Delta\varphi) \sim \left\{ \int \Psi_L^*(z)\Psi_R(z) + \text{c. c.} \right\}$$

This expression is proportional to $\cos(\Delta\varphi)$ and to the WKB factor $\exp(-\int_{-L/2}^{L/2}\sqrt{2m_p(2V_0-2\mu)}dz)$: since the integrand is twice that for single-electron tunneling, we have $f_{\text{WKB}} \sim |t|^2$. Thus,

$$E_J(\Delta\varphi) = A|t|^2 \cdot \cos(\Delta\varphi)$$

But the constant A is tricky! At first sight A>0, but this cannot be right, since the solution for $\Delta\varphi = 0$ ("Ψ_+") is nodeless, whereas that for $\Delta\varphi = \pi$ ("Ψ_-") has a node, so that Ψ_- must lie higher ($\mathrm{E}_- > \mathrm{E}_+$) For general gap $\Delta\varphi$

$$\Psi_{\Delta\varphi}(\mathrm{z}) \equiv \cos(\Delta\varphi/2)\Psi_+(\mathrm{z}) + i\sin(\Delta\varphi/2)\Psi_-(\mathrm{z})$$

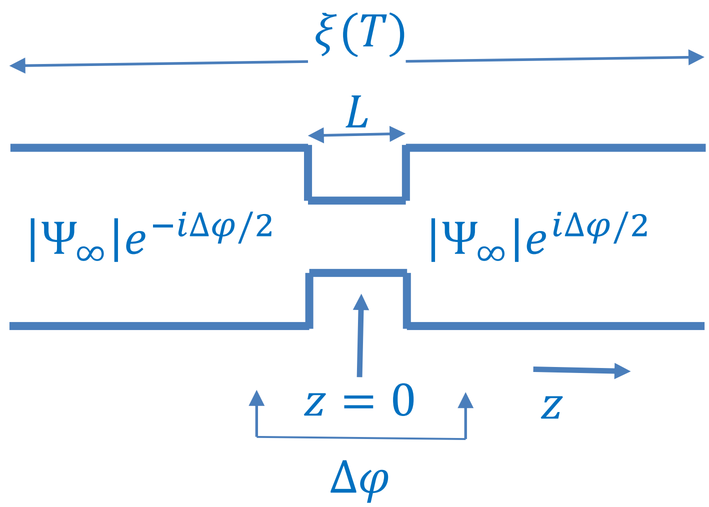
so that by linearity of the TISE ⟵ time-independent Schrodinger equation

$$E(\Delta\varphi) = (\text{const.}+)(E_+ - E_-)\cos(\Delta\varphi)) \equiv -E_J\cos(\Delta\varphi),$$
$$E_J \equiv E_- - E_+ > 0$$

A more detailed calculation confirms $E_J \propto |t|^2$: since the normal-state resistance R_N of the junction $\propto |t|^{-2}$ we have

$$E_J \propto R_N^{-1}$$

(b) More realistic example: "short" microbridge ($L \ll \xi(T)$)



In dimensionless form GL free energy is

$$F\{\Psi(\mathrm{z})\} = \mathcal{F}_0(T) \int \{-|f|^2 + \tfrac{1}{2}|f|^4 + \tfrac{1}{2}\xi^2(T)|\tfrac{df}{dz}|^2\} dz$$

bulk free energy $\qquad \Psi(z)/\Psi_\infty \qquad$ GL healing length

With boundary conditions $f(-\infty) = e^{-i\Delta\varphi/2}$, $f(+\infty) = e^{+i\Delta\varphi/2}$. If $\xi(T) \gg L$ and $\Delta\varphi \neq 0$, bending term will dominate, so we minimize it $\Rightarrow \partial^2 f/\partial z^2 = 0$.
Solution:

$$f = \frac{1}{2}\left\{\left(1 + \frac{z}{\frac{L}{2}}\right) e^{\frac{i\Delta\varphi}{2}} + \left(1 - \frac{z}{\frac{L}{2}}\right) e^{-\frac{i\Delta\varphi}{2}}\right\}$$

$$\equiv \cos(\Delta\varphi/2) + i(z/(L/2))\sin(\Delta\varphi/2)$$

Free energy is dominated by bending term, *i.e.* by the $\sin \Delta\varphi/2$ term:

$$\Delta F = \frac{2A}{L}\mathcal{F}_0(T)\xi^2(T)\sin^2\Delta\varphi/2$$ (A = cross-section)

or using $\mathcal{F}_0(T)\xi^2(T) = \hbar^2/(2m)|\Psi_\infty(T)|^2$

$$\Delta F = \frac{\hbar^2 A}{2mL}|\Psi_\infty|^2(1 - \cos\Delta\varphi)$$

so that

$$E_J = \frac{\hbar^2}{2mL}|\Psi_\infty|^2$$

Note again $E_J \propto R_N^{-1}$ (in sense that it $\propto A/L$)

<u>(c) Realistic (Bardeen-Josephson) model of tunnel junction</u>
(result only)

(Ambegaokar-Baratoff): start from single-electron tunnelling, express matrix elements in terms of Bogoliubov quasiparticles, then coherence factors [not discussed in these lectures] give nontrivial dependence on $\Delta\varphi$. Result at $T = 0$

$$\mathrm{I_c} = \frac{\pi\Delta}{\mathrm{eR_N}} \qquad \left(\equiv \frac{2\pi E_J}{\Phi_0}\right)$$

or at non-zero T

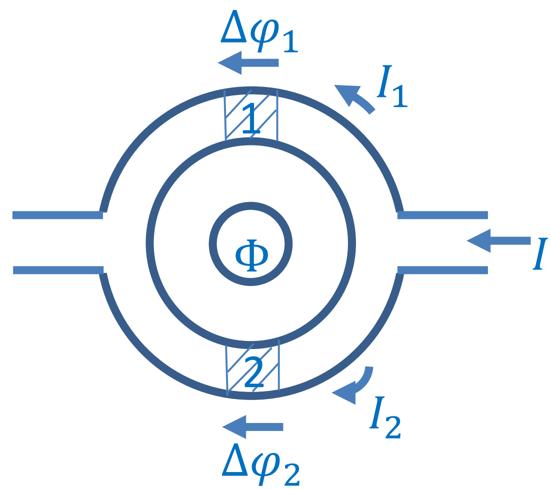
$$I_c = \left(\frac{\pi\Delta}{eR_N}\right)\tanh(\Delta(T)/2T)$$

AB formula

usually fairly well satisfied in junctions between "classic" superconductors

The dc SQUID

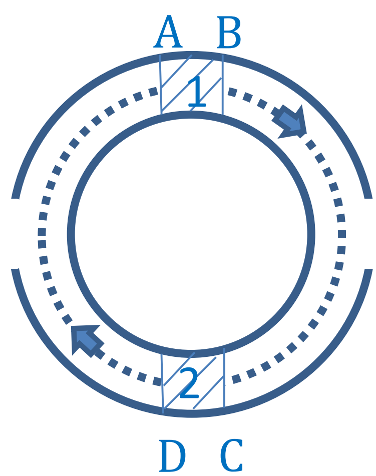
$$I = I_1 + I_2 = I_c(\sin\Delta\varphi_1 + \sin\Delta\varphi_2)$$



but, $\Delta\varphi_1$ and $\Delta\varphi_2$ are not independent! Analogously to above discussion of single ring,

$$\Delta\varphi_{AD} = \frac{2e}{\hbar}\int_D^A A \cdot dl$$

$$\Delta\varphi_{CB} = \frac{2e}{\hbar}\int_B^C A \cdot dl$$



so if contributions to $\int$ from junctions themselves negligible,

$$\Delta\varphi_1 - \Delta\varphi_2 = 2\pi\,\Phi/\Phi_0$$

Hence if $\xi \equiv \frac{1}{2}(\Delta\varphi_1 + \Delta\varphi_2)$,

$$I = I_c(\sin\Delta\varphi_1 + \sin\Delta\varphi_2) = 2I_c\sin\xi\cos(\pi\Phi/\Phi_0)$$

so total critical current of SQUID or $f(\Phi)$ (attained for $\xi = \pi/2$) is

$$I_c(\Phi) = 2I_c|\cos(\pi\Phi/\Phi_0)|$$

with max ($2I_c$) at $\Phi = n\Phi_0$ and min (zero) at $\Phi = (n + 1/2)\Phi_0$.
(Application to magnetometry - lecture 12)

An extension of the argument gives for a single junction subject to a parallel magnetic field

$$I_c(\Phi) = I_c \frac{|\sin(\pi\Phi/\Phi_0)|}{|\pi\Phi/\Phi_0|}$$

where
$\Phi = BLd_{eff}$ ⟵ $d + 2\lambda_L$