Quasiparticles in Normal and Superfluid Fermi Liquids

Tony Leggett
Department of Physics
University of Illinois Urbana-Champaign

partly joint work with Yiruo Lin

1. (Brief) review of Landau Fermi-liquid idea
2(a). Quasiparticles in the superfluid state: BCS theory
    Quasiparticles in the superfluid: Bogoliubov-de Gennes approach
3. Majarana fermions (orthodox approach)
4. Majarana fermions (doubts)

QUASIPARTICLES IN NORMAL PHASE

Landau (1956), Nozières, Theory of Interacting Fermi Systems (1964):

Start from noninteracting Fermi gas,

then energy eigenstates specified by $\{n(\boldsymbol{p},\sigma)\}, n(p,\sigma) = 0 \text{ or } 1$

groundstate has $n(\boldsymbol{p},\sigma) = \Theta(p_F - |\boldsymbol{p}|),\ p_F = \hbar(3\pi^2 n)^{1/3}$

switch on inter-particle interaction $\hat{V}$ adiabatically:

$$\hat{V}(t) = \hat{V}\exp(\alpha t) \qquad t < 0, \alpha \to 0\left(\hat{V}(t=0) = \hat{V}\right)$$

provided perturbation theory converges, states of fully interacting system can be labelled by the noninteracting states $\{n_0(\boldsymbol{p},\sigma)\}$ from which they evolved. ("adiabatic" evolution)

then define "no. of quasiparticles in state p,σ" $\mathrm{n}(\boldsymbol{p},\sigma)$ as simply equal to $\mathrm{n}_0(\boldsymbol{p},\sigma)$

(⇒ Luttinger theorem trivial)

(Actually, adiabaticity is sufficient but not necessary (S. Shastry, Ann. Phys. **405**, 155 (2019))

Suppose $\hat{Q} = \sum_{p\sigma} q\,(\boldsymbol{p}\sigma)\alpha^{+}_{p\sigma}\alpha_{p,\sigma}$

then in original noninteracting system $\hat{Q} = \sum_{p\sigma} q\,(\boldsymbol{p}\sigma) n_0(\boldsymbol{p},\sigma)$.

Is it true that in fully interacting system also $\hat{Q} = \sum_{p\sigma} q\,(\boldsymbol{p}\sigma) n(\boldsymbol{p},\sigma)$?

Answer: yes, if and only if $\left[\hat{Q}, \hat{H}(t)\right] = 0$

What quantities are conserved $\left(\left[\hat{Q},\hat{H}(t)\right]=0\right)$?

(a) liquid ^3He:

total number	$\hat{N}=\sum_{p\sigma}\alpha^+_{p\sigma}\,\alpha_{p\sigma}$	yes
total spin	$\hat{S}\equiv\sum_{p\sigma}\sigma\alpha^+_{p\sigma}\,\alpha_{p\sigma}$	yes
total current	$\hat{\boldsymbol{J}}=m^{-1}\sum_{p\sigma}\boldsymbol{p}\alpha^+_{p\sigma}\,\alpha_{p\sigma}$	yes
total spin current	$\boldsymbol{J}_\sigma\equiv m^{-1}\sum_{p\sigma}(\boldsymbol{p}\sigma)\alpha^+_{p\sigma}\,\alpha_{p\sigma}$	no

$\Rightarrow$ in real liquid ^3He,

$$\hat{S}=\sum_{p\sigma}\sigma n\,(\boldsymbol{p}\sigma)\qquad\text{(etc.)}$$

but $\hat{J}_\sigma\neq\sum_{p\sigma}\frac{\boldsymbol{p}}{m}\sigma n(\boldsymbol{p}\sigma)$

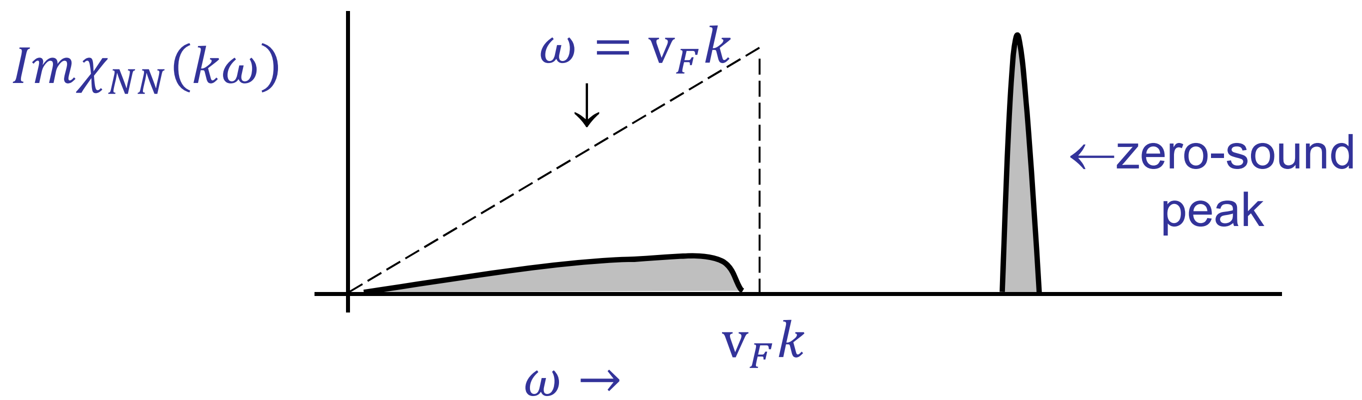
(b) metallic system (e.g. cuprates)

N, S conserved but $\boldsymbol{J}$ not conserved (even after transformation to Bloch states, because of U-processes).

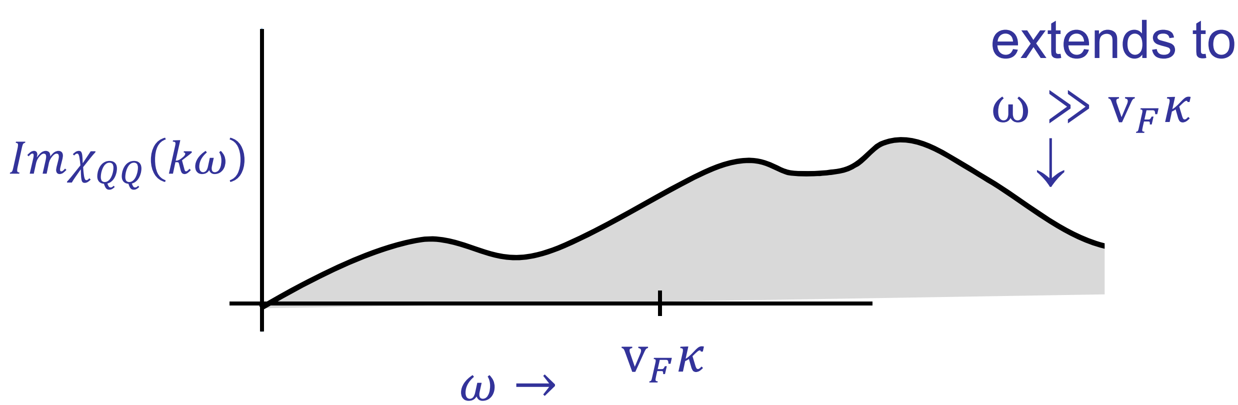
Consequences of conservation for response functions

Consider $\chi_{QQ}(k,\omega) \equiv FT$ of $\ll \hat{Q}(0,0)\hat{Q}(rt) \gg$

If $\hat{Q}$ is conserved, then in limit $k \to 0$, support of $Im\chi_{QQ}(k\omega)$ comes entirely from quasiparticle states and is limited to $\omega \lesssim \mathrm{v}_F k$
Ex: $\hat{Q} = \hat{N}$ (density response function) of liquid ^3He



If $\hat{Q}$ is not conserved, e.g. $\hat{Q} = \hat{J}_\sigma$ in ^3He, quasiparticle states do not exhaust sum rule for $\chi_{QQ}(k\omega)$ and even in limit $\mathrm{k} \to 0$ get incoherent background.



In liquid ^3He, can infer quasiparticle contribution to $\chi_{J_\sigma J_\sigma}$ from Landau parameter F_a^1 (measurable in spin-echo experiments).
Conclusion:

In ^3He, incoherent background contributes >80% of sum rule!

Moral: Even if system is a “decent” Fermi liquid, correlation function of non-conserved quantity can have large contribution from incoherent background.

Application to cuprates (and maybe other SCES):

N, S conserved but $\boldsymbol{J}$ not conserved (because of U-processes)

$\Rightarrow$ sum rule for $\omega Im\chi_{JJ}(\omega)$ $\left(= Re\ \sigma(\omega)\right)$ can have large contribtion from incoherent background (MIR peak).

Are the optimally doped and underdoped cuprates simply “bad” Fermi liquids?
(cf. e.g. Berthod et al., PR B **87**, 115109 (2013)).

QUASIPARTICLES IN SUPERFLUID STATE

1. BCS CASE: MEAN-FIELD TREATMENT

$$\hat{H}_{MF} = \sum_k \hat{H}(k) \Longrightarrow \Psi = \prod_k \Psi_k$$

$$\hat{H}(k) = \varepsilon_k (a^+_{k\uparrow} a_{k\uparrow} + a^+_{-k\downarrow} a_{-k\downarrow}) + (\Delta_k a^+_{k\uparrow} a^+_{-k\downarrow} + H.C.)$$

breaks U(1) symmetry

$\varepsilon_k = \left(\frac{\hbar^2 k^2}{2m} - \mu\right)$

Δ_k: to be determined self-consistently

Hilbert space of Ψ_k is 4D: $|0\ 0\rangle, |0\ 1\rangle, |1\ 0\rangle, |1\ 1\rangle$

even no. parity:

Ground pair state:

$$\Psi_k^{(GP)} = u_k|0\ 0\rangle + v_k|1\ 1\rangle \equiv (u_k + v_k a^+_{k\uparrow} a^+_{-k\downarrow})|\text{vac}\rangle$$

$$u_k \equiv \frac{1}{\sqrt{2}}\left(1 + \frac{\varepsilon_k}{E_k}\right), v_k \equiv \frac{1}{\sqrt{2}}\left(1 - \frac{\varepsilon_k}{E_k}\right),$$

$$E_k \equiv \left(\varepsilon_k^2 + |\Delta_k|^2\right)^{1/2}$$

Excited-pair state:

$$\Psi_k^{(EP)} = v_k|0\ 0\rangle - u_k|1\ 1\rangle, E_{EP} - E_{GP} = 2E_k$$

Odd-no-parity states (Bogoliubov quasiparticles):

Standard textbook maneuver: solve $[\hat{H}_{MF}, \hat{\Omega}] = E\hat{\Omega}$

with $\hat{\Omega} \equiv$ lin. combination of $a_{k\uparrow}^+, a_{-k\downarrow}^+, a_{k\uparrow}, a_{-k\downarrow}$

Bogoliubov quasiparticle creation operator

result: 2 solutions with $\mathrm{E} = +E_k$, e.g.

$$\Psi_{BP}^{(k)} = (u_k a_{k\Gamma}^+ - v_k a_{-k\downarrow})\, \left|\Psi_k^{(GP)}\right\rangle \equiv \alpha_{k\uparrow}^+ \left|\Psi_k^{(GP)}\right\rangle \equiv |1\ 0\rangle$$

2 solutions with $E = -E_k$, e.g.

$$\Psi_k^{(-)} = (v_k a_{k\uparrow}^+ + u_k a_{-k\downarrow})\, \left|\Psi_k^{(GP)}\right\rangle \equiv \beta_{k\uparrow}^+ \left|\Psi_k^{(GP)}\right\rangle$$

However,

vector of zero norm

$\Psi_k^{(-)} \equiv 0$, i.e. $\beta_{k\uparrow}^+$ is pure annihilator

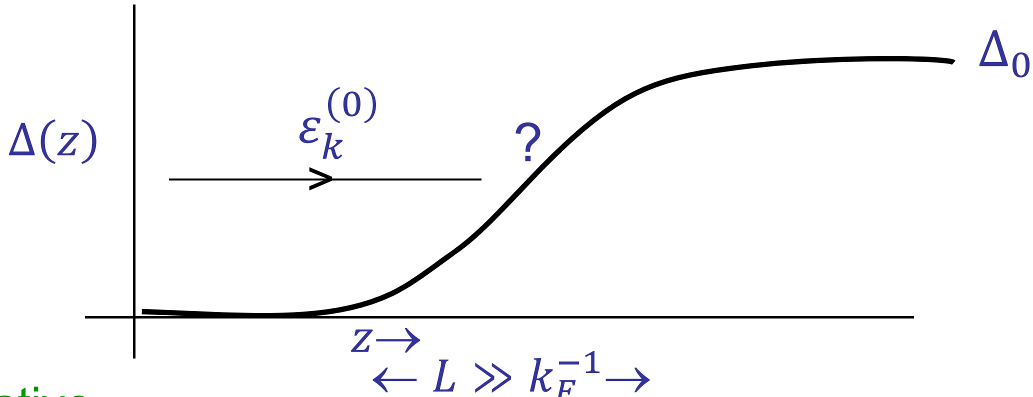
This is obvious since $\beta_{k\uparrow}^+$ is H.C. of second Bogoliubov quasiparticle creation operator $\alpha_{-k\downarrow}^+$.

(note: $[\hat{H}, \hat{\Omega}]|\Psi_{GP}\rangle = -E\hat{\Omega}|\Psi_{GP}\rangle$ is compatible with $\hat{\Omega}|\Psi_{GP}\rangle$=0 for any E!)

SOME PROPERTIES OF BOGOLIUBOV QUASIPARTICLES

1. reflection by barrier

e.g.:



energy relative to Fermi energy

Incident particle, $0 < \varepsilon_k < \Delta_0$

for normal reflection one needs $\Delta k \sim 2k_F$ so amplitude $\sim \exp - 2k_F L \ll 1$, and Fermi sea blocked:??

Quasiclassical discussion: $E = E(\boldsymbol{k}, \boldsymbol{r})$,

$$E(\boldsymbol{k}, \boldsymbol{r}) \equiv \left(\varepsilon_k^2 + |\Delta(r)|^2\right)^{1/2}$$

$$\hbar \frac{d\boldsymbol{k}}{dt} = -\frac{\partial E}{\partial \boldsymbol{r}}(\boldsymbol{k}, r), \qquad \frac{dr}{dt} = \frac{1}{\hbar}\frac{\partial E}{\partial \boldsymbol{k}} \cong \mathrm{v}_F(\mathcal{E}_k / E_k)$$

at point $\varepsilon_k^{(0)} = |\Delta(r)|, \varepsilon_k = 0 \Longrightarrow \boldsymbol{k}$ falls through Fermi surface $\Longrightarrow$

particle reflected as hole (Andreev reflection)

by conservation of energy, far from barrier $E_k = |\epsilon_k| \Rightarrow$ hole energy is $-\epsilon_k$.

What is momentum transfer in Andreev reflection?
$\epsilon_k \sim \hbar v_F (k - k_F) \Rightarrow$

$$\Delta p = \frac{2\epsilon_k}{v_F} \ll p_F$$ (normal incidence)

Is there direct experimental evidence for this? Yes!

Buchanan et al., (PRL 57, 341 (1986) measure terminal velocity of $^3He\ \ A - B$ interface.

$v_{term} = \Delta G_{AB}/\Gamma \leftarrow$ frictional force due to reflection of qps

If we assume reflection is "normal", $v_{term} \lesssim 1\ \mathrm{mm/sec}$

Experimentally, $v_{term} \sim 0.1 - 1\ \mathrm{m/sec} \Rightarrow$ Andreev reflection

(S-K Yip and AJL, PRL **57**, 345 (1986))

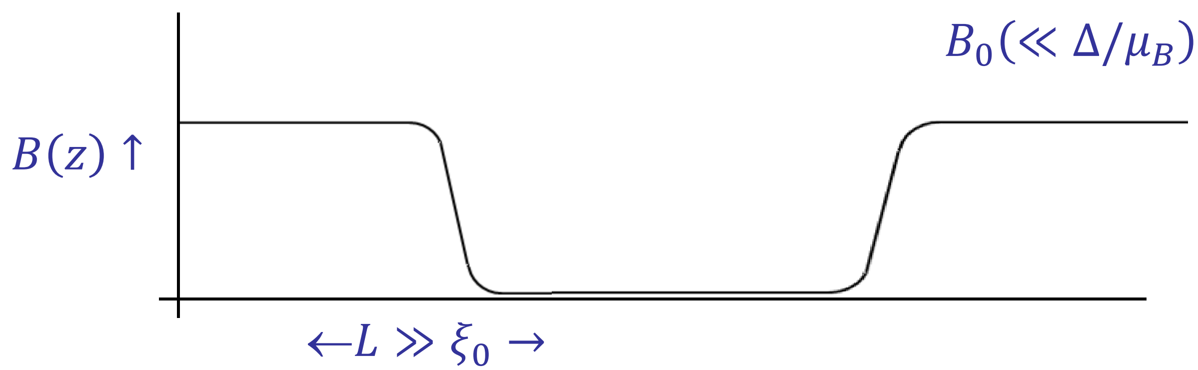
2. The "Zeeman-dimple" problem

Note: spatial variation of gap $\Delta(z)$ not a necessary condition for AR!

Andreev reflection

Can alternatively result from spatial variation of "diagonal" potential $V(r)$, provided this is the same for particle and hole (e.g. Zeeman potential $-\mu_B \sigma B(z)$)

Ex*: neutral Fermi superfluid with Zeeman coupling to external field $B(z)$ with "dimple"



Even (number) – parity ground state Ψ_0 has $\Delta(z) = const \equiv \Delta$ to linear order in B.

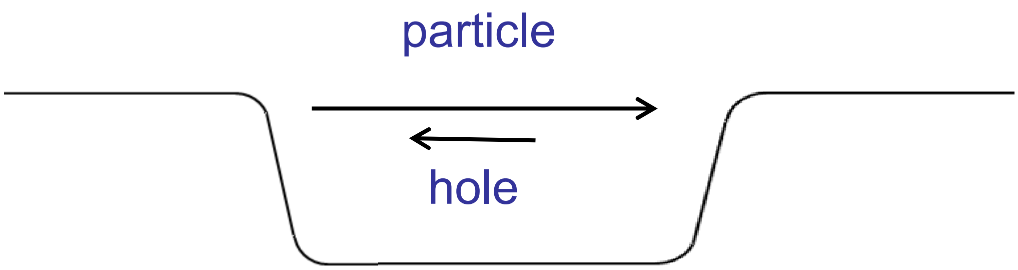
What is nature of lowest-energy odd-parity state?

Answer: Single Bogoliubov quasiparticle trapped in "dimple".

Extra spin localized in/close to dimple = 1.

*Y.-R. Lin and AJL, JETP **119**, 1034 (2014)

What is extra charge?



In quasiclassical approximation with only Andreev reflection:

$$E_{hole} = E_{particle} \Rightarrow \epsilon_{hole} = -\epsilon_{particle}$$

but in formula

$$\psi_{qp} = (u_k a^{+}_{k\uparrow} - v_k a^{+}_{-k\downarrow})|\Psi_0\rangle$$

$$u_k = \frac{1}{\sqrt{2}}(1 + \varepsilon_k/E_k), v_k = \frac{1}{\sqrt{2}}(1 - \varepsilon_k/E_k)$$

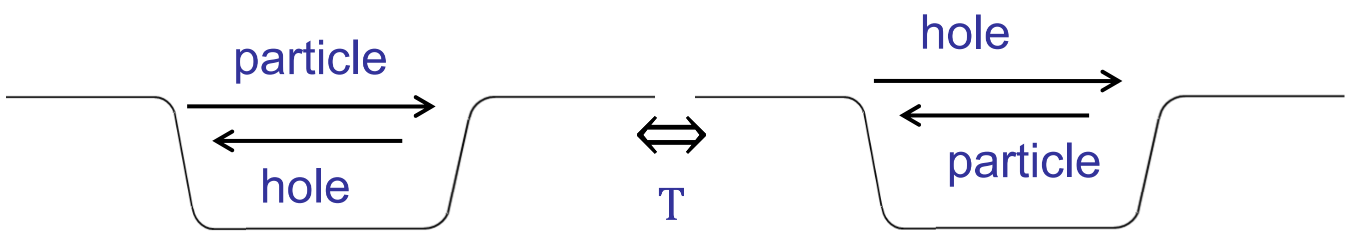
so $\varepsilon_k \to -\varepsilon_k \Rightarrow u_k \rightleftarrows v_k$. Also, $\text{velocity}_\text{k} = \hbar^{-1}\dfrac{\partial E_k}{\partial k} k$

$= \hbar^{-1}\,(\varepsilon_k/E_k)(\partial\varepsilon_k/\partial k)$, so to the extent that N-state spectrum is particle-hole symmetric, $(\partial\epsilon_k/\partial k = \hbar v_F = const.)$,

extra charge = 0

but this result is not “robust”.

Further complication: in this approximation, ground state of odd-number-parity sector is doublet related by time reversal!



"Normal" (non-Andreev) reflection splits doublet into even and odd combinations with exponentially small splitting. However, this does not change situation with regard to C-symmetry.

⇒ zero extra charge is not robust. (even in quasiclassical approximation)

Another interesting question: if geometry is that of torus and the "dimple" is dragged once around it, what is resultant Berry phase

(a) if superfluid is stationary

(b) if it is moving with $v_s = \hbar/2mR$ ("Abrikosov vortex")?

BEYOND BCS

In original BCS case, simple relation between even-number-parity states (Cooper pairs) and odd-number-parity ones (Bogoliubov quasiparticles). In the more general case this is no longer so.

Generic ansatz for (particle-conserving) completely paired GS of even-N system:

$$\Psi_N = n \cdot \left\{ \sum_{\alpha\beta} \int \int d\boldsymbol{r} d\boldsymbol{r}' \, \kappa_{\alpha\beta} (\boldsymbol{r}, \boldsymbol{r}') \psi_\alpha^\dagger(\boldsymbol{r}) \psi_\beta^\dagger(r') \right\}^{N/2} |\text{vac}\rangle$$

↑ normalization ↑ antisymmetric

By standard theorem (Yang 1962) can always rewrite in form

$$\Psi_N = n' \cdot \left(\sum_n c_n a_n^+ a_{\bar{n}}^+ \right)^{N/2} |\text{vac}\rangle$$

with $\{n\}, \{\bar{n}\}$ nonintersecting complete half-sets. If we break U(1) symmetry as in BCS, then even-parity GS is of form

$$\Psi_{\text{even}} = \prod_n \Psi_{n.}^{(GP)}$$

$$\Psi_n^{(GP)} = (u_n + v_n a_n^+ a_{\bar{n}}^+)|\text{vac}\rangle$$

(similar to BCS with $\boldsymbol{k} \uparrow \rightarrow n, -k \downarrow \rightarrow \bar{n}$

Also similarly to BCS, we can say

(a) "Excited pair" state is $(v_n - u_n a_n^+ a_{\bar{n}}^+)|\text{vac}\rangle$

(b) Operators $\mathrm{v}_n a_n^+ + u_n a_{\bar{n}}$, $v_n a_{\bar{n}}^+ - u_n a_n$ are pure annihilators

However:

the obvious guess at a Bogoliubov quasiparticle operator, namely

$$\alpha_n^+ = u_n a_n^+ - v_n a_n$$

indeed generates a state $|1\ 0\rangle$ which is an odd parity state, but

this state is in general not an eigenstate of the Hamiltonian!

In fact, we need to write the Bogoliubov quasiparticle states in form

$$\Psi_{\text{odd},i} = \sum_n (c_{in}|1\ 0\rangle_n + d_{in}|0\ 1\rangle_n) \equiv \sum_n (c_{in}\alpha_n^+ + d_{in}\alpha_{\bar{n}}^+)\left|\Psi_n^{(GP)}\right\rangle$$

with c_{in}, d_{in} fixed by minimizing MF Hamiltonian.

Bogoliubov-de Gennes ↘

In coordinate representation, mean-field (BdG) Hamiltonian is schematically of form

$$\hat{H}_{mf} = \sum_{\alpha\beta}\left\{\underbrace{\int dr\, K_{\alpha\beta}(r)\psi_\alpha^\dagger(r)\psi_\beta(r) + \frac{1}{2}\iint dr\, dr'\Delta_{\alpha\beta}(\boldsymbol{r},\boldsymbol{r}')\psi_\alpha^\dagger(r)\psi_\beta^\dagger(r')}_{\text{bilinear in } \psi_\alpha(r),\, \psi_\alpha^\dagger(r)\text{:}} + HC\right\}$$

(with a term $\mu\delta_{\alpha\beta}$ included in $K_{\alpha\beta}(r)$ to fix average particle number $\langle\hat{N}\rangle$.) In our context, interesting problem is to find simplest fermionic (odd-parity) states ("Bogoliubov quasiparticles"). For this purpose write schematically $\Psi_i = \gamma_i^+\Psi_{\text{even}}$, where (ignoring (real) spin degree of freedom)

$$\hat{\gamma}_i^\dagger = \int\{u_i(r)\hat{\psi}^\dagger(r) + v_i(r)\hat{\psi}(r)\}\,dr \qquad \left(\equiv \begin{pmatrix} u(r) \\ v(r) \end{pmatrix}\right)$$

←"Nambu spinor"

and determine the coefficients $u_i(r), v_i(r)$ by solving the Bogoliubov-de Gennes equations

$$[\hat{H}_{mf}, \hat{\gamma}_i^\dagger] = E_i\hat{\gamma}_i^\dagger$$

so that

$$\hat{H}_{mf} = \sum_i E_i\gamma_i^\dagger\gamma_i + \text{const.}$$

(All this is standard textbook stuff…)

Note crucial point: In mean-field treatment, fermionic quasiparticles are quantum superpositions of particle and hole ⇒ do not correspond to definite particle number (justified by appeal to SBU(1)S).* This "particle-hole mixing" is sometimes (misleadingly) regarded as analogous to the mixing of different bands in an insulator by spin-orbit coupling. (hence, analogy "topological insulator" ⇄ topological superconductor.)

Another note: just as in simple BCS case, "negative-energy" solutions of the BdG equations are fictitious (they simply correspond to operations which annihilate the GS).

*spontaneously broken U(1) symmetry

Majorana fermions

Recap: fermionic (Bogoliubov) quasiparticles created by operators

$$\gamma_i^\dagger = \int dr\,\{u_i(r)\hat{\psi}^\dagger(r) + v_i(r)\hat{\psi}(r)\}$$

with the coefficients $u_i(r), v_i(r)$ given by solution of the BdG equations

$$[\hat{H}_{mf}, \gamma_i^\dagger] = E_i\gamma_i$$

Question: Do there exist solutions of the BdG equations such that

$\gamma_i^\dagger = \gamma_i$ (and thus $E_i = 0$)?

This requires (at least)

1. Spin structure of $u(r), v(r)$ the same ⇒ pairing of parallel spins (spinless or spin triplet, not BCS s-wave)

2. $u(r) = v^*(r)$

3. "interesting" structure of $\Delta_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{r}')$, e.g. "$p + ip$" $\left(\Delta(\boldsymbol{r}, \boldsymbol{r}') \equiv \Delta(\boldsymbol{R}, p) \sim \Delta(R)(p_x + ip_y)\right)$

Case of particular interest: "half-quantum vortices" (HQV's) in Sr_2RuO_4 (widely believed to be $(p + ip)$ superconductor). In this case a M.F. predicted to occur in (say) ↑↑ component, (which sustains vortex), not in ↓↓ (which does not). Not that vortices always come in pairs (or second MF solution exists on boundary)

Why the special interest for topological quantum computing?

(1) Because MF is exactly equal superposition of particle and hole, it should be undetectable by any local probe.

(2) MF's should behave under braiding as Ising anyons*: if 2 HQV's, each carrying a M.F., interchanged, phase of MBWF changed by $\pi/2$ (note not π as for real fermions!)

So in principle‡:

(1) create pairs of HQV's with and without MF's

(2) braid adiabatically

(3) recombine and "measure" result

$\Downarrow$

(partially) topologically protected quantum computer!

* D. A. Ivanov, PRL **86**, 268 (2001)
‡ Stone & Chung, Phys. Rev. B **73**, 014505 (2006)

Comments on Majarama fermions (within the standard "mean-field"approach)

(1) What is a M.F. anyway?

Recall: it has energy exactly zero, that is its creation operator $\gamma_i^\dagger$ satisfies the equation

$$[H, \gamma_i^\dagger] = 0$$

But this equation has two possible interpretations:

(a) $\gamma_i^\dagger$ creates a fermionic quasiparticle with exactly zero energy (i.e. the odd- and even-number-parity GS's are exactly degenerate)

(b) $\gamma_i^\dagger$annihilates the (even-parity) groundstate ("pure annihilator")

However, it is easy to show that in neither case do we have $\gamma_i^\dagger = \gamma_i$. To get this we must superpose the cases (a) and (b), i.e.

a Majarana fermion is simply a quantum superposition of a real Bogoliubov quasiparticle and a pure annihilator.

But Majorana solutions always come in pairs ⇒ by superposing two MF's we can make a real zero-energy fermionic quasiparticle

HQV 1 HQV 2

$\gamma_1^\dagger$ $\gamma_2^\dagger$

Bog. qp. ⟶ $\alpha^\dagger \equiv \gamma_1^\dagger + i\gamma_2^\dagger$

The curious point: the extra fermion is "split" between two regions which may be arbitrarily far apart! (hence, usefulness for TQC)

↖ topological quantum computing

Thus, e.g. interchange of 2 vortices each carrying an MF ⟹ rotation of zero-energy fermion by π. (note predicted behavior (phase change of $\pi/2$) is "average" of usual symmetric (0) and antisymmetric (π) states)

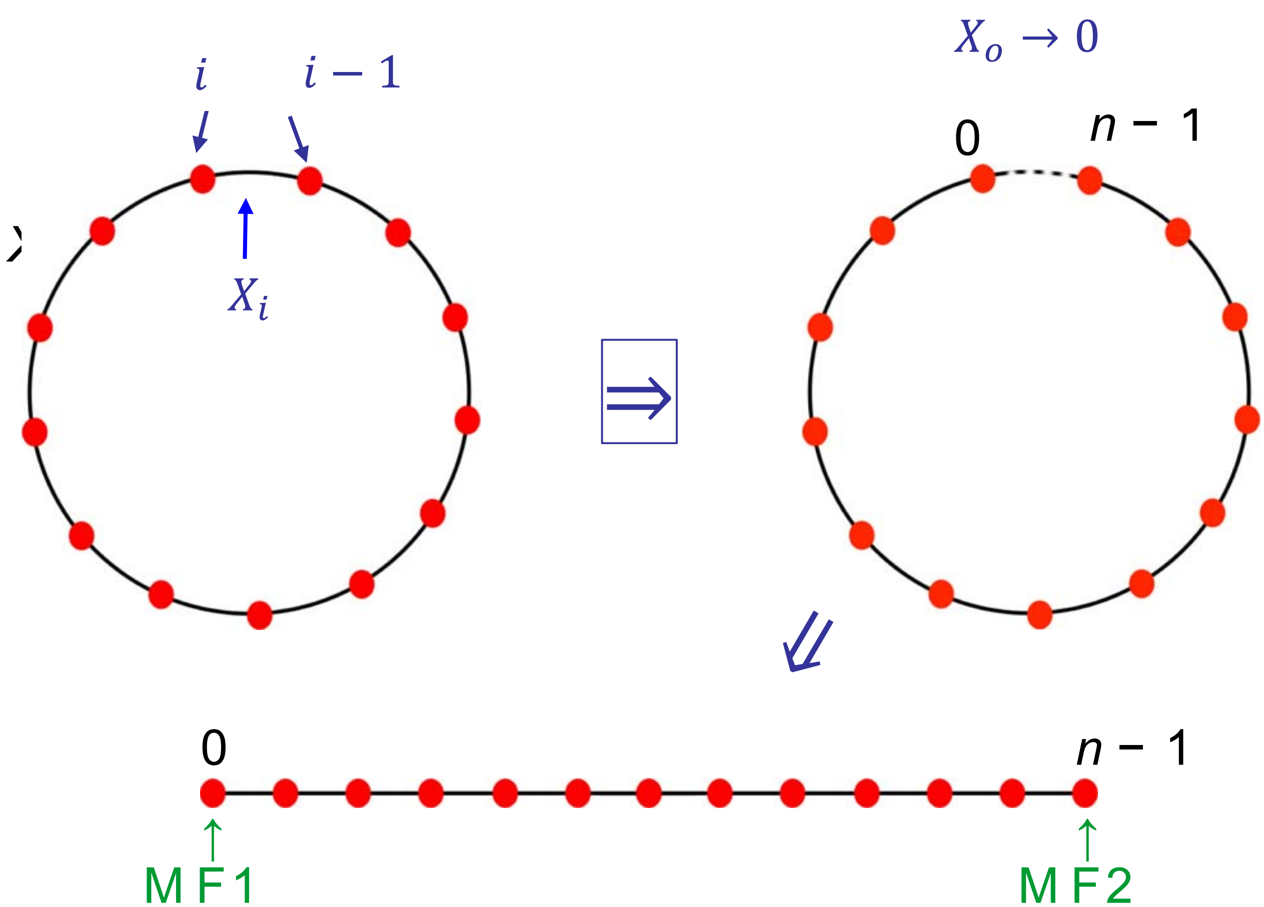
An intuitive way of generating MF's in the KQW:

Kitaev quantum wire

For this problem, fermionic excitations have form

$$\alpha_i^\dagger = (a_i^\dagger + i a_i) + (a_{i-1}^\dagger + i a_{i-1})$$

so localized on links not sites. Energy for link $(i, i-1)$ is X_i



Majorana fermions: beyond the mean-field approach

Problem: The whole apparatus of mean-field theory rests fundamentally on the notion of SBU(1)S ← spontaneously broken U(1) gauge symmetry:

$$\Psi_{\text{even}} \sim \sum_{\substack{N= \\ \text{even}}} C_N \, \Psi_N \qquad \left(C_N \sim |C_N| e^{iN\varphi}\right)$$

$$\Psi_{\text{odd}} \sim \int dr \left\{u(r)\hat{\psi}^\dagger(r) + v(r)\hat{\psi}(r)\right\} |\Psi_{\text{even}}\rangle \; \left(\equiv \hat{\gamma}_i^\dagger |\Psi_{\text{even}}\rangle\right) \; *$$

But in real life condensed-matter physics,

SB U(1)S IS A MYTH!!

This doesn't matter for the even-parity GS, because of "Anderson trick":

$$\Psi_{2\text{N}} \sim \int \Psi_{\text{even}}(\varphi) \exp -iN\varphi \, d\varphi$$

But for odd-parity states equation (*) is fatal! Examples:

(1) Galilean invariance

(2) NMR of surface MF in ^{3}He-B

We must replace (*) by

creates extra Cooper pairs

$$\hat{\gamma}_i^{\dagger} = \int dr \left\{ u(r)\hat{\psi}^{\dagger}(r) + v(r)\hat{\psi}C^{\dagger} \right\}$$

This doesn't matter, so long as Cooper pairs have no "interesting" properties (momentum, angular momentum, partial localization...)

But to generate MF's, pairs must have "interesting" properties!

⇒ doesn't change arguments about existence of MF's, but completely changes arguments about their braiding, undetectability etc.

Need completely new approach!