`Virasoro Algebra` for the Strange Metal

thanks to

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NSF
strange metal: experimental facts

Quadratic critical behaviour in a high-\( T_c \) superconductor

\[ \rho_{ab} \propto T \]

\[ L_{xy} = \kappa_{xy}/T \sigma_{xy} \neq \# \propto T \]

\[ \sigma(\omega) = C \omega^{-\frac{2}{3}} \]

\[ \frac{n\tau e^2}{m} \frac{1}{1 - i\omega \tau} \]
Theories of cuprates
Theories of strange metal = ∞

why is the problem hard?
single-parameter scaling

\[ \xi \Rightarrow \xi_T \propto \xi^z \]

\[
\frac{\delta^2 \ln Z}{\omega \delta A_\mu \delta A_\mu} \rightarrow 1
\]

\[
\rho \propto T^{(2-d)/z}
\]

\[
\sigma(\omega, T) \propto \omega^{(d-2)/z} \rightarrow -2/3
\]

\[
C_v \propto T^{d/z}
\]

**anomalous dimension**

\[ 2 \rightarrow 2d_A \]
dimension of vector potential

A

µ

A

µ

+ \partial_\mu G

has no units

\[[A_\mu] = \frac{1}{L} = 1\]

vector potential cannot have an anomalous dimension

conserved current
dimensions of current

\[ \phi(x) \rightarrow \phi(x) + \delta \phi(x) \]

\[ [J_0(x), \phi(y)] = \delta \phi(x) \delta^d(x - y) \]

\[ [J_0] = d \]

\[ \nabla \cdot J_\mu = \partial_t J_0 \]

\[ \xi_\tau \approx \xi \]

\[ [J_\mu] = d \]
can the dimension of the current change?

\[ \nabla \cdot J_\mu = \partial_t J_0 \]

\[ [J_\mu] = d - \theta + z - 1 \]

\[ d \rightarrow d - \theta \]

allowable by spacetime symmetries

\[ [A_\mu] = 1 \]
Hall Angle

\[
\cot \theta_H \equiv \frac{\sigma_{xx}}{\sigma_{xy}} \approx T^2
\]

Hall Lorenz ratio

\[
L_{xy} = \frac{\kappa_{xy}}{T \sigma_{xy}} \neq \# \propto T
\]

all explained if

\[
[J_\mu] = d - \theta + \Phi + z - 1
\]

Hartnoll/Karch

\[
[A_\mu] = 1 - \Phi
\]

\[
\Phi = -\frac{2}{3}
\]
current algebra with an anomalous dimension?

is there a direct experimental probe of this anomaly?

how can A have an anomalous dimension?

what else can be explained with the anomalous dimension?
what else can be explained with the anomalous dimension?
Drude conductivity

\[
\frac{n \tau e^2}{m} \frac{1}{1 - i\omega \tau}
\]

\[
\sigma(\omega) = C \omega^{\gamma-2} e^{i\pi(1-\gamma/2)}
\]

\[
\gamma = 1.35
\]
take experiments seriously

\[ \sigma^i(\omega) = \frac{n_i e_i^2 \tau_i}{m_i} \frac{1}{1 - i\omega \tau_i} \]

continuous mass (scale invariant unparticles)

\[ \sigma(\omega) = \int_0^M \frac{\rho(m) e^2(m) \tau(m)}{m} \frac{1}{1 - i\omega \tau(m)} dm \]
variable masses for everything

\[ \rho(m) = \rho_0 \frac{m^{a-1}}{M^a} \]

\[ e(m) = e_0 \frac{m^b}{M^b} \]

\[ \tau(m) = \tau_0 \frac{m^c}{M^c} \]

Karch, 2015

\[ \sigma(\omega) = \frac{\rho_0 e_0^2 \tau_0}{M^{a+2b+c}} \int_0^M dm \frac{m^{a+2b+c-2}}{1 - i\omega \tau_0 \frac{m^c}{M^c}} \]

perform integral
\[ \frac{a + 2b - 1}{c} = -\frac{1}{3} \]

\[ \sigma(\omega) = \frac{\rho_0 e_0^2 \tau_0^{\frac{1}{3}}}{M} \frac{1}{\omega^{\frac{2}{3}}} \int_0^{\tau_0} dx \frac{x^{-\frac{1}{3}}}{1 - ix} \]

\[ \omega \tau_0 \to \infty \]

\[ \sigma(\omega) = \frac{1}{3} (\sqrt{3} + 3i) \pi \frac{\rho_0 e_0^2 \tau_0^{\frac{1}{3}}}{M \omega^{\frac{2}{3}}} \]

\[ \tan \sigma = \sqrt{3} \]

\[ 60^\circ \]
are anomalous dimensions necessary

\[
\frac{a + 2b - 1}{c} = -\frac{1}{3}
\]

\[
\rho(m) = \rho_0 \frac{m^{a-1}}{M^a}
\]

\[
e(m) = e_0 \frac{m^b}{M^b}
\]

\[
\tau(m) = \tau_0 \frac{m^{c}}{M^c}
\]

\[
c = 1
\]

\[
a + 2b = \frac{2}{3}
\]

\[
b = 0
\]

\[
a = \frac{2}{3}
\]
No

but the Lorenz ratio is not a constant

\[ L_H = \frac{\kappa_{xy}}{T \sigma_{xy}} \sim T \equiv T^{-2\Phi/z} \]

Hartnoll/Karch

\[ \Phi = b z = -2/3 \]

AC+DC require anomalous dimension

\[ \rho \propto T^{-2\Phi/z} \]
how can A have an anomalous dimension?

\[ A_\mu \to A_\mu + \partial_\mu G \]

\[ F_{\mu\nu} = \partial_{[\mu} A_{\nu]} \]

\[ \nabla^\alpha \cdot J_\mu = \partial_t J_0 \]

both J and A have anomalous dimensions!
what kind of an operator is $\partial^\alpha_\mu$
\[ V \text{ open set} \]

if \( f(x) = 0, \, x \in V \)

and \( \hat{T} f(x) = 0, \, x \in V \)

for only \( f = 0 \)

then \( \hat{T} \) is antilocal

\[ \partial_\alpha \gamma^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} \gamma^{\beta - \alpha} \quad \rightarrow \quad \partial_\alpha \gamma^\mu = \frac{\gamma^{-\alpha}}{\Gamma(1 - \alpha)} \]

\[ \partial_\mu \text{ anti-local operator} \]
$\hat{T}$ is antilocal

$\hat{T} f(x)$

$f(x)$
physical consequences of anomalous dimension for $A_\mu$

$$A_\mu \rightarrow A_\mu + \partial_\mu \mathcal{G}$$

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$$

$$\tilde{\nabla}^\alpha \times \vec{A} = \vec{B}$$

no Stokes’ theorem

$$\oint \vec{A} \cdot d\ell \neq \int_S \vec{B} \cdot d\vec{S}$$

Aharonov-Bohm Effect must change
physical gauge connection

\[ a_i \equiv [\partial_i, I_i^\alpha A_i] = \partial_i I_i^\alpha A_i \]

\[ A_\mu \to A_\mu + \partial^\alpha_\mu \Lambda \]

\[ a_\mu \to a_\mu + \partial_\mu \Lambda \]

\[-\frac{\hbar^2}{2m} (\partial_i - i \frac{e}{\hbar} a_i)^2 \psi = i\hbar \partial_t \psi.\]
compute AB phase

\[ \Delta \Phi = \frac{e}{\hbar} \int \vec{a}(r') \cdot \vec{d}l' \]

use fractional calculus

\[ \Delta \phi_R = \frac{eBl^2}{\hbar} \left( \frac{b^{\alpha-1}d^{\alpha-1}}{\Gamma^2(\alpha)} \right) c \gg l, d \gg l \]
\[ \Delta \phi_D = \frac{e}{\hbar} \pi r^2 B R^{2\alpha - 2} \left( \frac{\sqrt{\pi} 2^{1-\alpha} \Gamma(2 - \alpha) \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\alpha) \Gamma\left(\frac{3}{2} - \frac{\alpha}{2}\right)} \sin^2 \frac{\pi \alpha}{2} \right)_2 F_1 \left( 1 - \alpha, 2 - \alpha; 2; \frac{r^2}{R^2} \right) \]
is the correction large?

\[ \alpha = 1 + \frac{2}{3} = \frac{5}{3} \]

\[ \Delta \Phi_R = \frac{eB\ell^2}{\hbar} L^{-5/3}/(0.43)^2 \]

yes!
Is this just a game?

Is there a consistent algebra for fractional currents?
Yes
Virasoro algebra

\[ L_n := -z^{n+1} \frac{\partial}{\partial z} \]

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} \]

Witt algebra

conformal transformations on unit disk

\[ \mathcal{V} \rightarrow \mathcal{W} \rightarrow 1 \]

central extension
Fractional Virasoro algebra

generators

\[ L_a^n = -z^{a(n+1)} \left( \frac{\partial}{\partial z} \right)^{a} \]

\[ \bar{L}_a^n := -\bar{z}^{a(n+1)} \left( \frac{\partial}{\partial \bar{z}} \right)^{a} \]

\[ [L_n, L_m](z^{ak}) = \left( \frac{\Gamma(a(k+n)+1)}{\Gamma(a(k-1+n)+1)} - \frac{\Gamma(a(k+m)+1)}{\Gamma(a(k-1+m)+1)} \right) L_{n+m}(z^{ak}) \]

\[ = \left( A_{a,n,m}^1(k) \otimes L_{n+m} \right)(z^{ak}) \]

\[ 1 \to \mathcal{H} \to \mathcal{V}_a \to \mathcal{W}_a \to 1 \]

\[ A_{n,m}^1(k) = n - m \]
Fractional Virasoro algebra

cohomology group

\[ H^2(\mathcal{W}_a, \mathcal{H}) = Z^2(\mathcal{W}_a, \mathcal{H})/B^2(\mathcal{W}_a, \mathcal{H}) \]

algebra for conformal non-local actions
origin of non-local actions
$g(z = 0, x) = f(x)$

$\gamma = \frac{1 - a}{2}$

$\Delta_x g + \frac{a}{z} \partial_z g + \partial_z^2 g = 0$

$g(x, 0) = f(x)$

use Caffarelli-Silvestre extension theorem (2006)
\[ \frac{1}{2} (\partial_\mu \phi)^2 + m^2 \phi^2 \]

\[ \mathcal{O} = C_\mathcal{O} \lim_{z \to 0} z^{-\Delta} \phi(x, z) \]

\[ g = z^{\gamma-d/2} \phi \]

\[ \phi \]

solves massive scalar problem

solves CS extension problem
exact form of boundary operator

\[ O = C_\partial \lim_{z \to 0} z^{-\Delta} \phi(x, z) \]

\[ = C_\partial \lim_{z \to 0} z^{-\Delta+1} \partial_z \phi(x, z) \]

\[ O = (-\Delta)^\gamma \phi_0 \]

use Caffarelli/Silvestre

the \( O \) for massive scalar field
application: gauge fields with anomalous dimensions

\[ F_{\mu \nu} F^{\mu \nu} + m^2 A_y^2 \]

\[ A_{\mu} \left( -\Delta \right)^\gamma A_{\mu}^{\perp} \]

\[ \gamma = \sqrt{d^2 + m^2 - 1/2} \]

dynamical `Higgs' mode

additional length scale
strange metal

combine AC+DC transport

\[ [J] = \Phi \]

\[ [A_\mu] = d_A \neq 1 \]

fraction not d

non-local action

probe by Aharonov-Bohm effect on underdoped cuprates

new Virasoro algebra