Noether’s Second Theorem and Strange Metals

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T-linear resistivity

\( \rho > \rho_Q \)

\( l < a \)

\( l > a \)

\( \rho > \rho_Q \)

Violates MIR limit

electrons not charge carriers

?
Is a new gauge principle operative?
Does anything local carry the charge?

- No
  - e is scale dependent
  - $\int_{\ell} eA \notin h\mathbb{Z}$

- Yes
  - $\int_{\ell} eA = h\mathbb{Z}$
Can you mess with A?
Noether’s First Theorem

\[ U(1) \quad \leftrightarrow \quad qA \rightarrow qA - q\partial_\mu \Lambda \]

\[ \psi' = e^{iqA} \psi \]

\[ [q\Lambda] = 0 \]
\[ [qA] = 1 \]
\[ [A] = 1 \]

\[ S = \int d^d x (J_\mu A^\mu + \cdots) \]

\[ \partial_\mu J^\mu = 0 \]

\[ [d^d x JA] = 0 \]

\[ [J] = d - 1 \]

fixes dimension of current

current conservation

Noether’s Thm. I
Are there exceptions?
Pippard’s problem

\[ J_s \neq \frac{-c}{4\pi \lambda^2} A \]  

failure of local London relations
\[ J_s = -\frac{3}{4\pi c \xi_0 \lambda} \int \frac{(\vec{r} - \vec{r}'')((\vec{r} - \vec{r}') \cdot \vec{A}(\vec{r}'')) e^{-(\vec{r} - \vec{r}'')/\xi(\ell)}}{(\vec{r} - \vec{r}'')^4} d^3 \vec{r}'' \]
Units of Current

\[ J_{\mu}(x) = \frac{\delta L_m}{\delta A_\mu} = - \int C^{\mu\nu}(x, x')(A_\nu(x') - \partial_\nu \phi(x')) d^3 x' \]

\[[J] = d - d_C - d_A\]

anomalous dimension

Standard Result

\[ \delta(x_0 - y_0)[J_{\mu}(x), \phi(y)] = \delta^d(x - y)\delta\phi(y) \]

\[[J] = d - 1\]
Are there other examples of currents with anomalous dimensions?

underlying electricity and magnetism?

is symmetry breaking necessary?
Mott insulator

- temperature
- parent
- under doped
- optimally doped
- over doped
- doping

- “strange” metal
- no order
- conventional metal?
why is the problem hard?
single-parameter scaling

$$\xi \downarrow \xi_T \propto \xi^z$$

$$\frac{\delta^2 \ln Z}{\omega \delta A_\mu \delta A_\mu}$$

$$\sigma(\omega, T) \propto \omega$$

$$C_v \propto T^{d/z}$$

anomalous dimension $2 \rightarrow 1$
strange metal explained!

Hall Angle
\[ \cot \theta_H \equiv \frac{\sigma_{xx}}{\sigma_{xy}} \approx T^2 \]

Hall Lorenz ratio
\[ L_{xy} = \frac{\kappa_{xy}}{T \sigma_{xy}} \neq \# \propto T \]

all explained if
\[ [J_\mu] = d - \theta + \Phi + z - 1 \]
\[ [A_\mu] = 1 - \Phi \]
\[ \Phi = -\frac{2}{3} \]

Hartnoll/Karch
\[ [J_\mu] = d - \theta + \Phi + z - 1 \]

\[ [A_\mu] = 1 - \Phi \]
\[ \Phi = -2/3 \]

\[ [E] = 1 + z - \Phi \]
\[ [B] = 2 - \Phi \]

\textbf{note:} \[ \pi r^2 B \neq \text{flux} \]

\[ \oint A \cdot dl \notin h\mathbb{Z} \]

?
How is this possible - - if at all?
what is the new gauge principle?

\[ [A_\mu] \neq 1 \]

\[ A_\mu \rightarrow A_\mu + \partial_\mu \Phi \]
Noether’s Second Theorem: precursor

**hint**

\[ \partial_\mu J_\mu = 0 \]  
**current conservation**

**what if**

\[ [\partial_\mu, \hat{Y}] = 0 \]

**new current**

\[ [\tilde{J}] = d - 1 - D_Y \]

**gauge symmetry**

\[ \partial_\mu \hat{Y} J_\mu = \partial_\mu \tilde{J}_\mu = 0 \]
possible gauge transformations

\[ S = \frac{1}{4} \int d^d x F^2 \]

\[ S = \frac{1}{2} \int \frac{d^d k}{2\pi^d} A_\mu(k) [k^2 \eta^{\mu\nu} - k^\mu k^\nu] A_\nu(k) \]

\[ M_{\mu\nu} k^\nu = 0 \]

zero eigenvector

\[ ik_\mu \rightarrow \partial_\nu \]

\[ A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \]
family of zero eigenvalues

\[ M_{\mu\nu} f k^\nu = 0 \]

generator of gauge symmetry

1.) rotational invariance
2.) A is still a 1-form
3.) \([f, k_\mu] = 0\)
only choice

\[ f \equiv f(k^2) \]

\[ (-\Delta)^\gamma \]

\[ A_\mu \rightarrow A_\mu + (-\Delta)^{\frac{\gamma - 1}{2}} \partial_\mu \Lambda \]

\[ [A_\mu] = \gamma \]

what kind of E&M has such gauge transformations?
claim: extra dimension

\[ S = \int dV_d dy \left( y^a F^2 + \cdots \right) \]

\[ \text{eom} \quad d(y^a \star dA) = 0 \]

\([A] \neq 1\]

Karch:1405.2926

Gouteraux: 1308.2084
if holography is RG then how can it lead to an anomalous dimension?
construct `boundary’ theory explicitly
\[ S = \int dV d\gamma (y^\alpha F^2 + \cdots) \]

\[ d(y^\alpha \star dA) = 0 \]
Caffarelli-Silvestre extension theorem (2006)

\[ g(x, y = 0) = f(x) \]
\[ \Delta_x g + \frac{a}{y} g_y + g_{yy} = 0 \]
\[ \nabla \cdot (y^a \nabla g(x, y)) = 0 \]

\[ \lim_{y \to 0} y^a \partial_y g \]

\[ C_{d, \gamma}(-\Delta)^\gamma f \]

\[ g(z = 0, x) = f(x) \]
\[ \gamma = \frac{1 - a}{2} \]
Closer look

\[ \nabla \cdot (y^a \nabla u) = 0 \]  
scalar field (use CS theorem)

\[ d(y^a \star dA) = 0 \]  
holography

Similar equations

generalize CS theorem to p-forms

GL, PP: 1708.00863
(CIMP, 366, 199 (2019)))
\[ d(\ast \rho^a dA) = 0 \]

UV
conformal boundary
\[ r \to \infty \]

\[ \rho = r - r_h \]
horizon
IR

\[ d(\ast y^a dA) = 0 \]

\[ A \to A + d\gamma \Lambda \equiv A' \]
\[ d\gamma \equiv \Box^{\frac{\gamma-1}{2}} d \]
boundary action: fractional Maxwell equations

\[ \Box^\gamma A_\perp = J \]

boundary action has `anomalous dimension' (non-locality)

\[ F \rightarrow d_\gamma A = \partial_\mu \Box^{(\gamma^{-1})/2} A_\nu - \partial_\nu \Box^{(\gamma^{-1})/2} A_\mu \]
if holography is RG then how can it lead to an anomalous dimension?

\[ S = \int dV_d dy \left( y^\alpha F^2 + \cdots \right) \]

\[ [A] = 1 - \alpha/2 \]

dimension of A is fixed by the bulk theory: not really anomalous dimension
$[\partial_\mu, \hat{Y}] = 0$

$[d, \Box \gamma] = 0$

$J \rightarrow \Box \gamma J \quad [J] = d - 1 - \gamma$
Ward identities

\[ C^{ij}(k) \propto (k^2)^\gamma \left( \eta^{ij} - \frac{k^i k^j}{k^2} \right). \]

standard Ward identity

\[ k_i C^{ij}(k) = 0 \quad \Rightarrow \quad \partial_i C^{ij}(k) = 0 \]

but

\[ k^{\gamma-1} k_\mu C^{\mu\nu} = 0 \quad \Rightarrow \quad \partial_\mu (-\Delta)^{\gamma-1/2} C^{\mu\nu} = 0 \]

inherent ambiguity in E&M
family of zero eigenvalues

\[ M_{\mu\nu} f k^\nu = 0 \]

most fundamental conservation law

\[ \partial^\mu (-\nabla^2)^{(\gamma-1)/2} J_\mu = 0 \]
Noether’s Second Theorem

\[
\sum_{\mu} \delta u_{\mu} = \delta f - \frac{d}{dx} \left\{ \sum \left( \begin{array}{c} 1 \\ \alpha \\
(1) \\
\end{array} \right) \frac{\partial f}{\partial u_{\alpha}} \delta u_{\alpha} + \left( \begin{array}{c} 2 \\ \alpha \\
(2) \\
\end{array} \right) \frac{\partial f}{\partial u_{\alpha}} \delta u_{\alpha}^{(2)} + \cdots + \left( \begin{array}{c} \kappa \\ \alpha \\
(\kappa) \\
\end{array} \right) \frac{\partial f}{\partial u_{\alpha}} \delta u_{\alpha}^{(\kappa-1)} \right\} + \\
+ \frac{d^2}{dx^2} \left\{ \sum \left( \begin{array}{c} 2 \\ \alpha \\
(2) \\
\end{array} \right) \frac{\partial f}{\partial u_{\alpha}} \delta u_{\alpha} + \left( \begin{array}{c} 3 \\ \alpha \\
(3) \\
\end{array} \right) \frac{\partial f}{\partial u_{\alpha}} \delta u_{\alpha}^{(3)} + \cdots + \left( \begin{array}{c} \kappa \\ \alpha \\
(\kappa) \\
\end{array} \right) \frac{\partial f}{\partial u_{\alpha}} \delta u_{\alpha}^{(\kappa-2)} \right\} + \\
\vdots \\
+ (-1)^{\kappa} \frac{d^\kappa}{dx^\kappa} \left\{ \sum \left( \begin{array}{c} \kappa \\ \alpha \\
(\kappa) \\
\end{array} \right) \frac{\partial f}{\partial u_{\alpha}} \delta u_{\alpha} \right\}
\]

\( A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \Lambda + \partial_{\mu} \partial_{\nu} G^{\nu} + \cdots \),

\( A \rightarrow A + d_{\gamma} \Lambda \equiv A' \)

\( d_{\gamma} \equiv (-\Delta)^{\frac{\gamma-1}{2}} d \)
Noether’s Second Theorem and Ward Identities for Gauge Symmetries

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For simplicity, we focus on the case when the transformation may be written in the form\textsuperscript{6}

\[
\delta_\lambda \phi = f(\phi) \lambda + f^\mu(\phi) \partial_\mu \lambda, \tag{10}
\]

but it is straightforward to consider transformations, as Noether did, involving arbitrarily high derivatives of \(\lambda\). (Although, the authors know of no physically interesting examples.) Let us start with

arxiv:1510.07038
experiments?
The magnetic flux \( \vec{B} \) should be dimensionless.

\[ \pi r^2 B \]

What's the resolution?

\[ [B] = 2 - \Phi = 2 + 2/3 \neq 2 \]
correct dimensionless quantity

\[ a_i \equiv [\partial_i, I_i^\alpha A_i] = \partial_i I_i^\alpha A_i \]

\[ \Delta^{-\alpha} \]

what's the relationship?

\[ \oint \partial\Sigma a = \oint \partial\Sigma A \]

Norm \[
\oint \partial\Sigma a = \frac{1}{\Gamma(3/2 - \gamma)} \oint \partial\Sigma A
\]

not an integer
obstruction theorem to charge quantization (NST)

\[ A_\mu \rightarrow A_\mu + \partial_\mu \Lambda + \partial_\mu \partial_\nu G^\nu + \cdots , \]

\[ A \rightarrow A + d_\gamma \Lambda \equiv A' \]

\[ d_\gamma \equiv (-\Delta)^{\gamma-1} \frac{1}{2} d \]

charge ill-defined (new landscape problem)
\[ \Delta \phi_D = \frac{e}{\hbar} \pi r^2 B R^{2\alpha - 2} \left( \frac{\sqrt{\pi} 2^{1-\alpha} \Gamma(2 - \alpha) \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\alpha) \Gamma\left( \frac{3}{2} - \frac{\alpha}{2} \right)} \sin^2 \frac{\pi \alpha}{2} _2F_1(1 - \alpha, 2 - \alpha; 2; \frac{r^2}{R^2}) \right) \]
is the correction large?

\[ \alpha = 1 + \frac{2}{3} = \frac{5}{3} \]

\[ \Delta \Phi_R = \frac{eB\ell^2}{\hbar} L^{-5/3} / (0.43)^2 \]

yes!
if in the strange metal

\[ [A_\mu] = d_A \neq 1 \]

God said…

\[ \Box^{d-1} \left( \nabla \times \vec{B} - \frac{1}{v^2} \frac{\partial \vec{E}}{\partial t} \right) = \mu \vec{j} \]
\[ \Box^{d-1} \nabla \cdot \vec{E} = \frac{\rho}{\epsilon} \]
\[ \Box^{d-1} \left( \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) = 0 \]
\[ \Box^{d-1} \nabla \cdot \vec{B} = 0. \]

fractional E&M

\[ J^\mu(x) = - \int d^d x' C_{\mu\nu}(|x - x'|) A^\nu \]

\[ [J] \neq d - 1 \]
\[ [A] \neq 1 \]

in SC!

\[ \omega = ck \]

\[ U(1) \neq \mathbb{Z}_2 \]