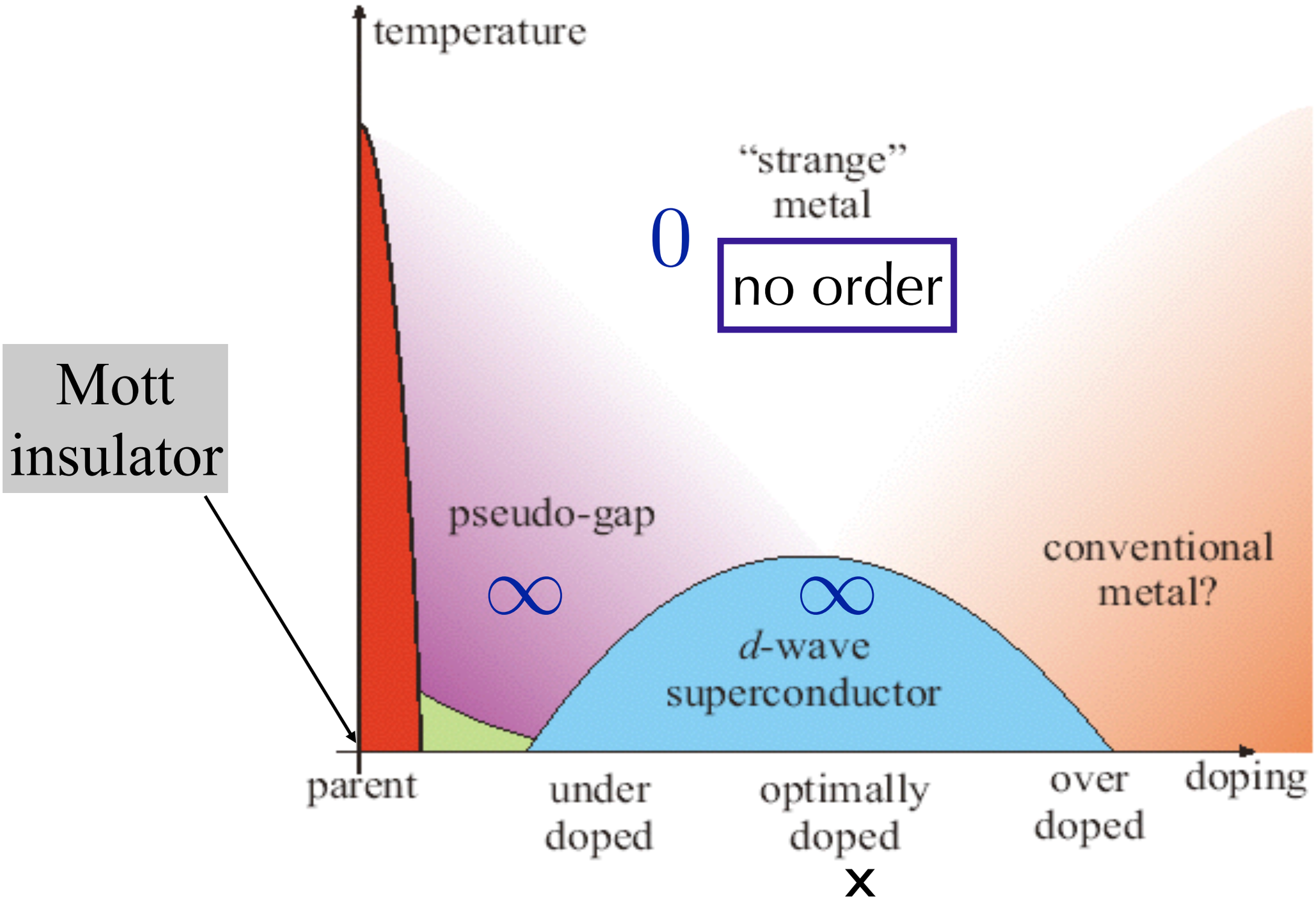
# `Anomalous dimensions' for conserved currents and Noether's Second Theorem

Thanks to: NSF, EFRC (DOE)

Gabriele La Nave

Kridsangaphong Limtragool


temperature
"strange"
metal
0
no order
Mott
insulator
pseudo-gap
∞
∞
conventional
metal?
d-wave
superconductor
parent
under
doped
optimally
doped
over
doped
doping
x

# strange metal: experimental facts

**Quantum critical behaviour in a high-$T_c$ superconductor**

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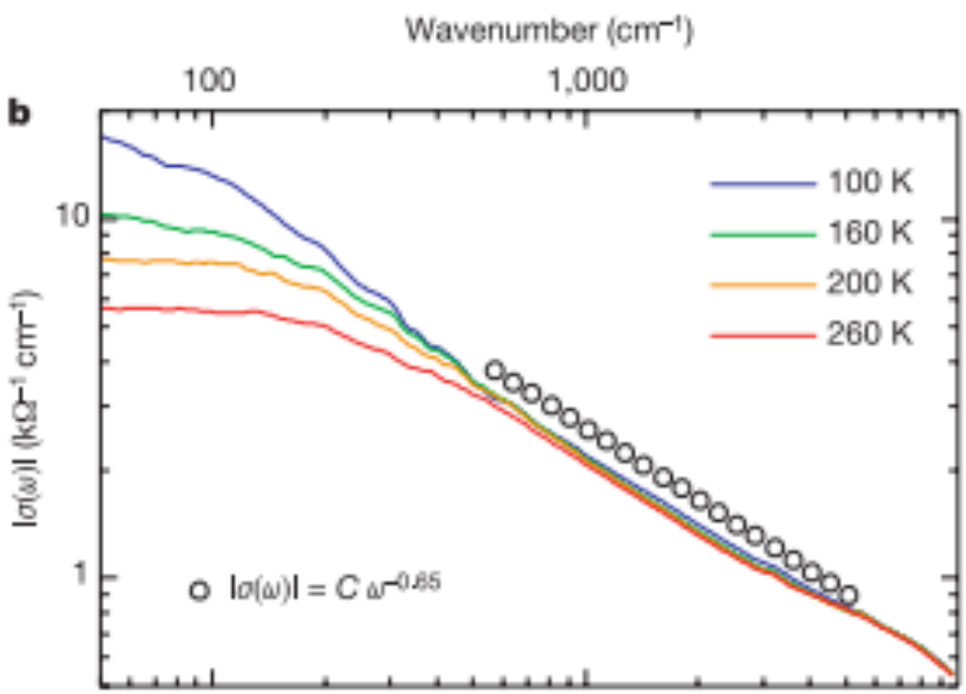
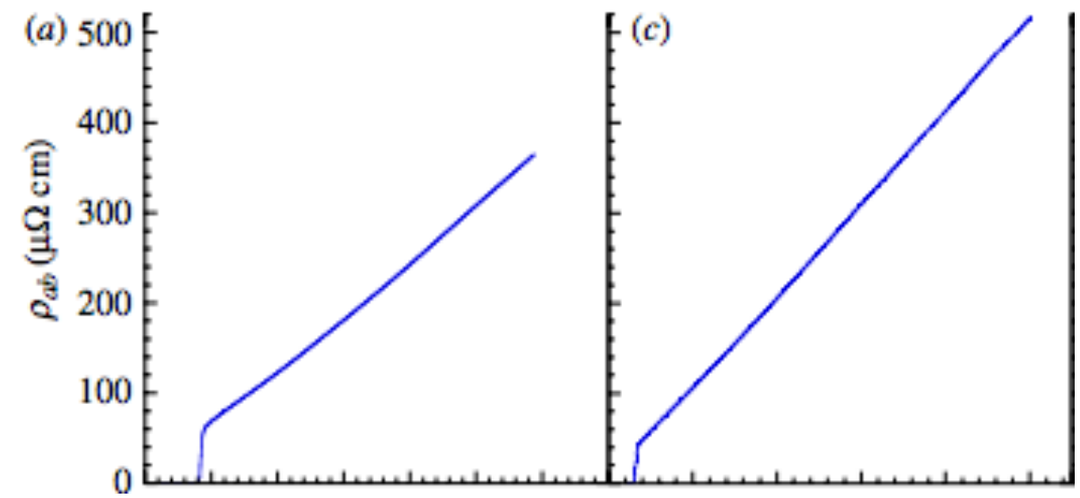


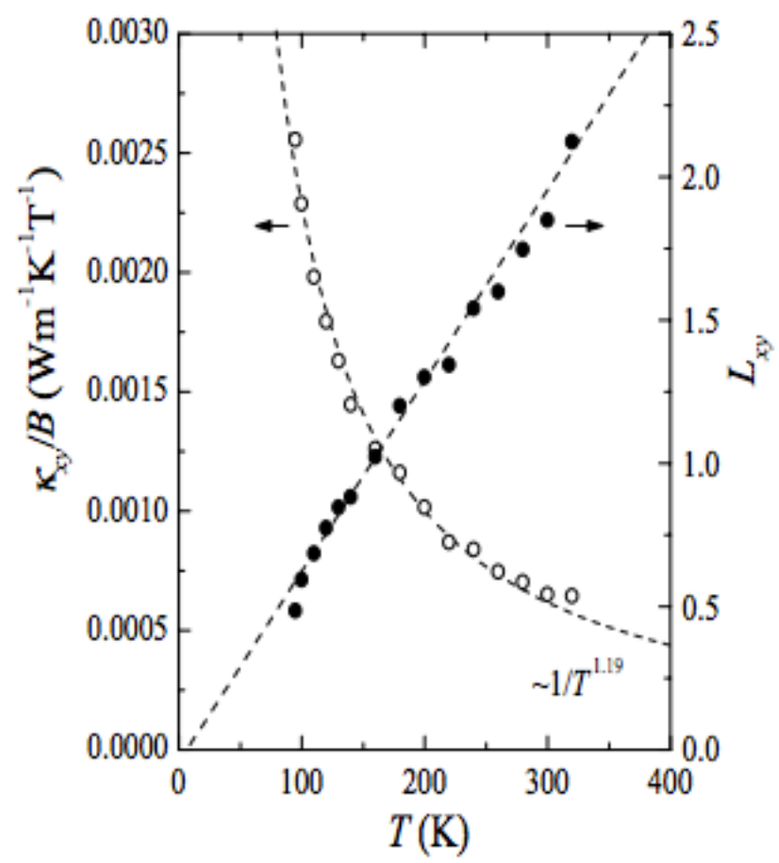

$$\sigma(\omega) = C\omega^{-\frac{2}{3}}$$

$$\frac{n\tau e^2}{m}\frac{1}{1-i\omega\tau}$$

## T-linear resistivity




$$L_{xy} = \kappa_{xy}/T\sigma_{xy} \neq \# \propto T$$



standard metals

resistivity

$$\rho \propto T^{2}$$

Weidemann-Franz law

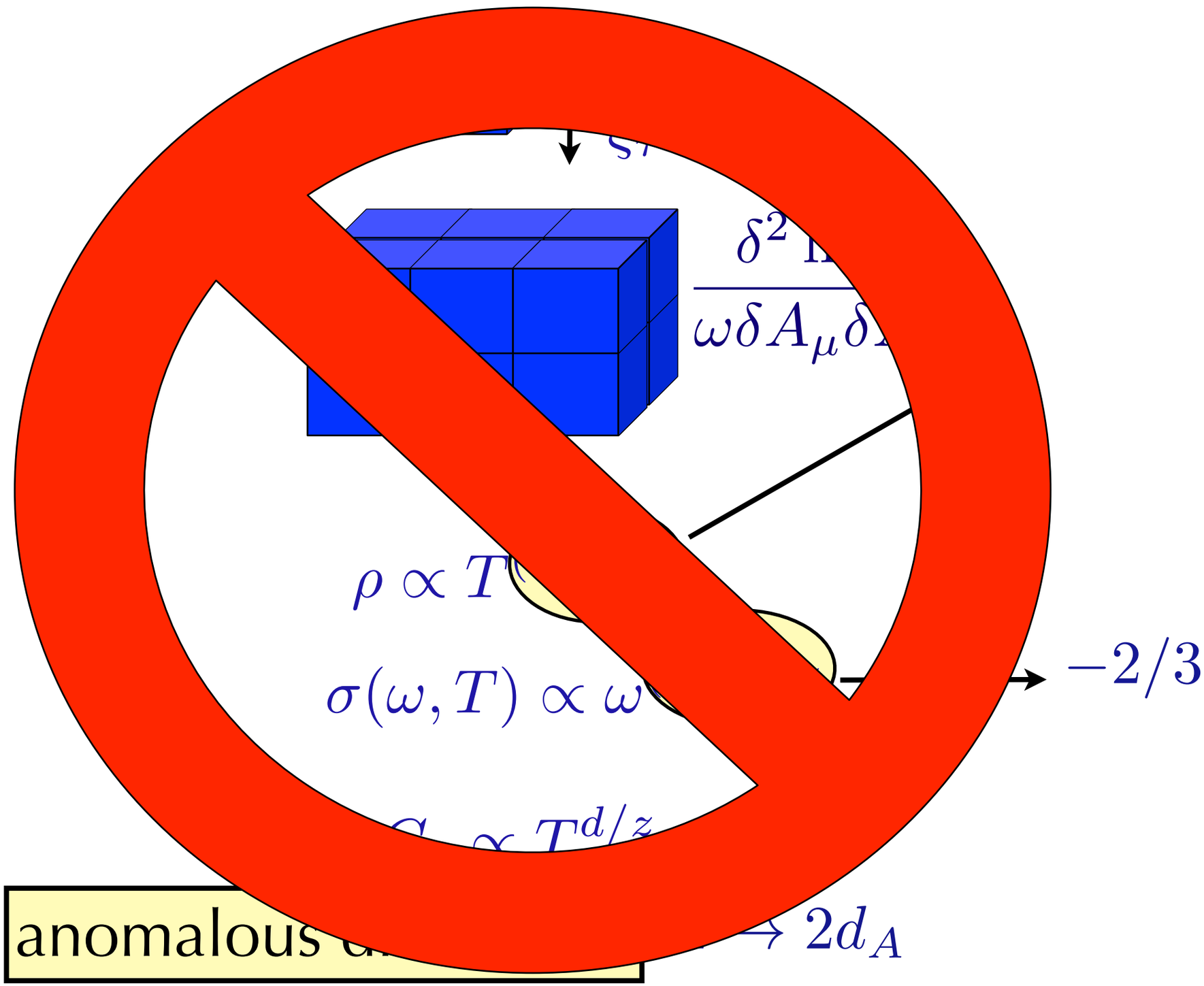
$$\frac{\kappa_{xx}}{T\sigma_{\mathrm{xx}}} = \frac{\pi^2}{3}$$

optical conductivity

$$\Re\sigma \propto 1/\omega^{2}$$

why is the problem hard?

single-parameter scaling




anomalous

dimension of vector potential

gauge invariance

$$\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \mathcal{G}$$

has no units

$$[A_\mu] = \frac{1}{L} = 1$$

vector potential cannot have an anomalous dimension

conserved current

gauge invariance= current conservation

$$S = \int d^d x \left( F^2 + J_\mu A^\mu + \cdots \right)$$

$$A \to A + \partial\Lambda$$

$$S \to S + \int d^d x \cancel{J_\mu \partial\Lambda}$$

integrate by parts

Noether's Thm. I

$$\partial_\mu J^\mu = 0$$

current conservation

gauge invariance

$$[A_\mu] = 1$$

current conservation

$$[d^d x J A] = 0$$

fixes dimension of current

$$[J] = d - 1$$

# general argument

$$\phi(x) \to \phi(x) + \delta\phi(x)$$

$$[J_0(x), \phi(y)] = \delta\phi(x)\delta^d(x-y)$$

$$\Downarrow$$

$$[J_0] = d$$

# photon propagator

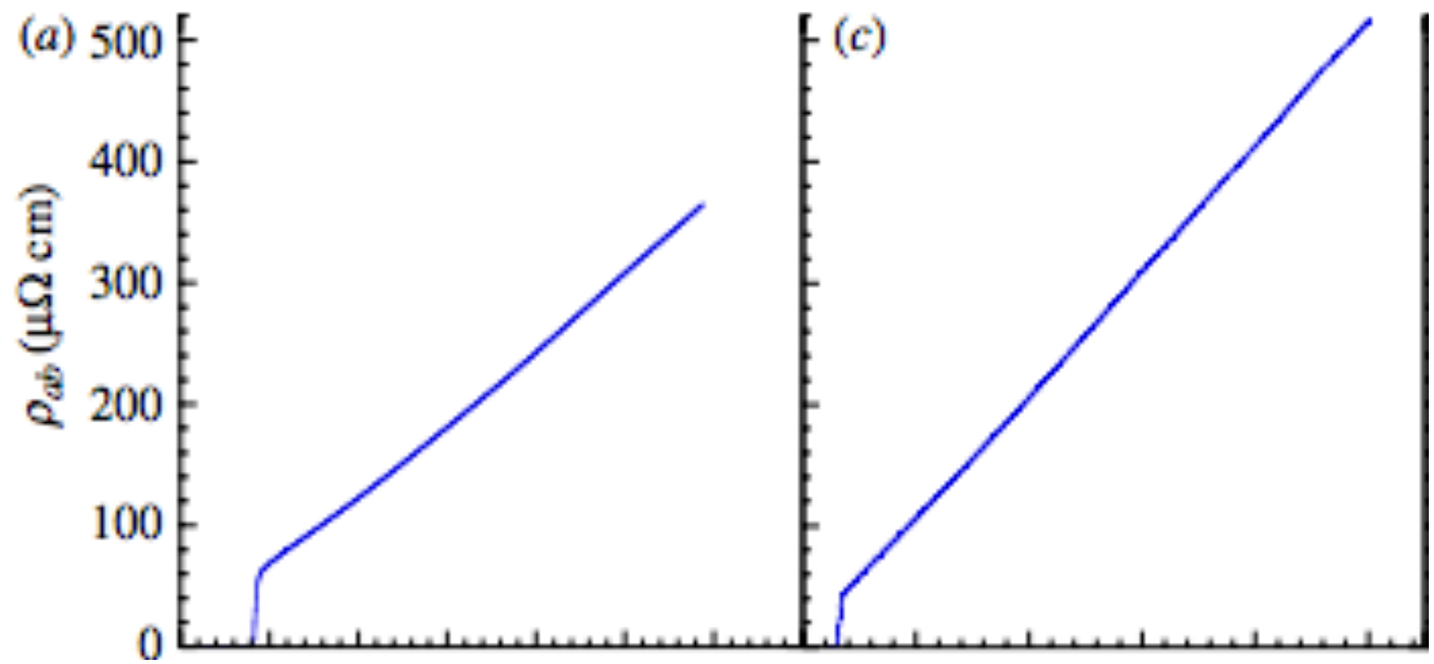
$$\langle A^i(k) A^j(-k) \rangle \propto \frac{\eta^{ij}}{k^2}$$

# strange metal explained!

## Hall Angle

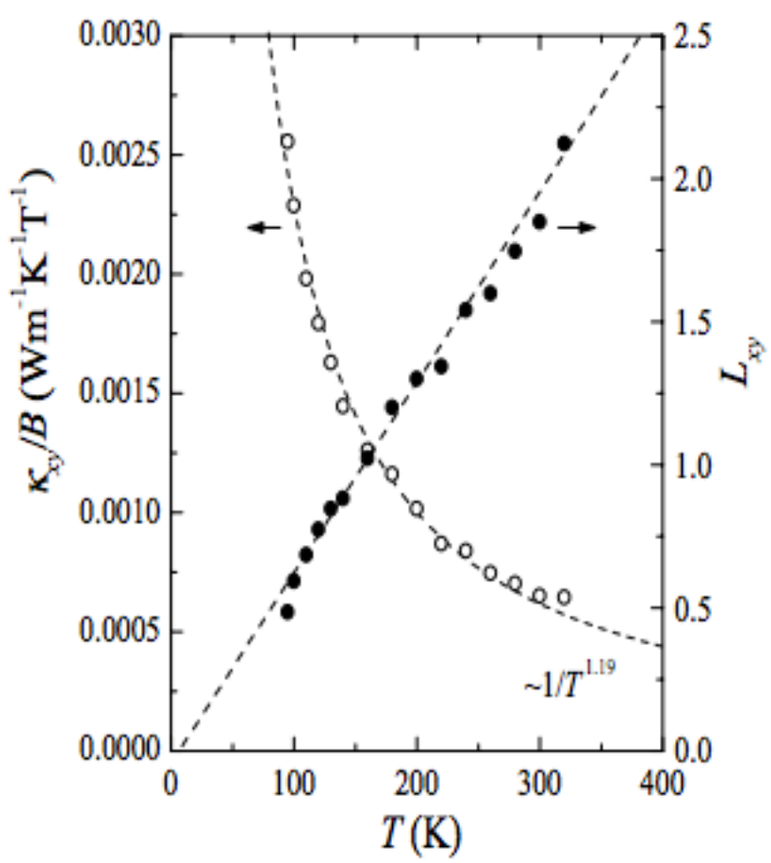
$$\cot\theta_H \equiv \frac{\sigma_{xx}}{\sigma_{xy}} \approx T^2$$

## T-linear resistivity




## Hall Lorenz ratio

$$L_{xy} = \kappa_{xy}/T\sigma_{xy} \neq \# \propto T$$




## all explained if

$$[J_\mu] = d - \theta + \Phi + z - 1$$

$$[A_\mu] = 1 - \Phi$$

$$\Phi = -2/3$$

## Hartnoll/Karch

# strange metal

$$[J_\mu] = d - \theta + \Phi + z - 1$$

$$[A_\mu] = 1 - \Phi$$

$$\Phi = -2/3$$

$$[E] = 1 + z - \Phi$$

$$[B] = 2 - \Phi$$


note $\pi r^2 B \neq$ flux

$$\langle A^i(k) A^j(-k) \rangle \propto \frac{\eta^{ij}}{k^{2(1-\Phi)}}$$

non-local propagator

# Scaling theory of conserved current and universal amplitudes at anisotropic critical points

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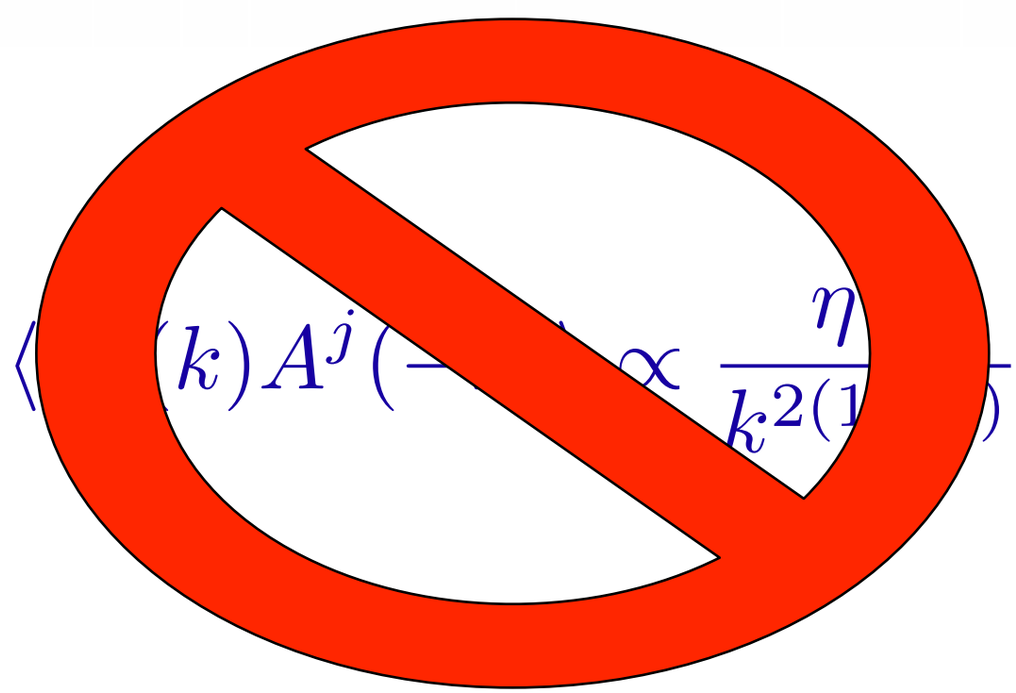
(Received 21 January 1992)

A scaling theory of conserved currents is studied at critical points with critical charge fluctuations. It is shown that the conserved current has no anomalous dimension even for anisotropic critical points. Many universal amplitudes are identified. One of the universal amplitudes at the critical points is

$$\lim_{(k,\omega)\to 0}\omega^{2-d}k^{(z-d)(2-d)}[\sigma(\omega)]^{z}[\kappa(k)]^{2-d},$$

which reduces to the universal conductance in 2+1 dimensions. We find that the ratio $\sqrt{\kappa\chi}/\sigma$ always approaches a constant as we approach the critical point. We also determine exponents of many scaling relations, using the properties of the current.



How is this possible - - if at all?

current algebra with an anomalous dimension?

is there a direct experimental probe of this anomaly?

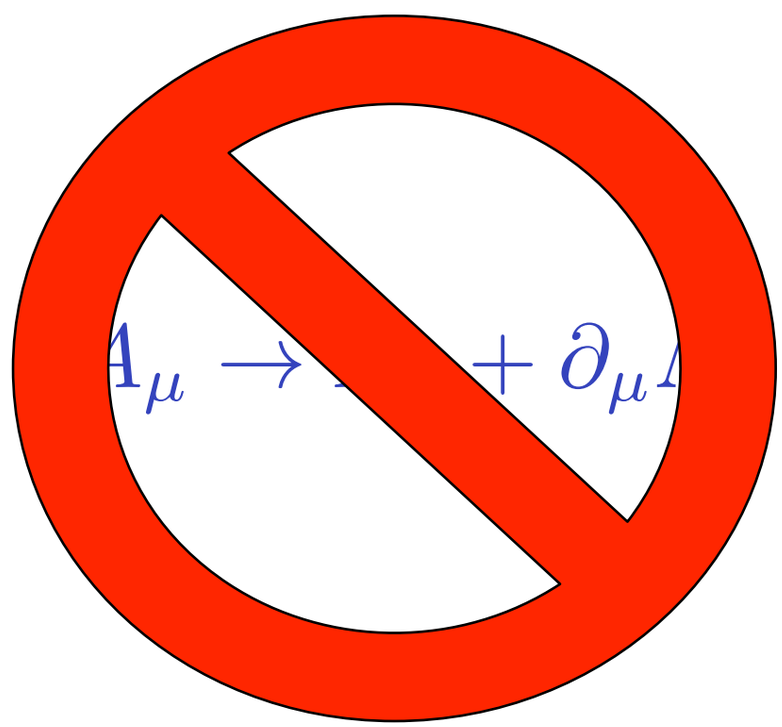
how can A have an anomalous dimension?

what else can be explained with an anomalous dimension?

what is the new gauge principle?

if

$$[A_\mu] \neq 1$$



hint

$$\partial_\mu J^\mu = 0$$

current conservation

what if

$$[\partial_\mu, \hat{Y}] = 0$$

new current

$$[\tilde{J}] = d - 1 - D_Y$$

$$\partial_\mu \hat{Y} J^\mu = \partial_\mu \tilde{J}^\mu = 0$$

possible gauge transformations

$$S = -\frac{1}{4}\int d^d x F^2$$

$$S = \frac{1}{2}\int \frac{d^d k}{2\pi^d} A_\mu(k)[k^2 \underbrace{\eta^{\mu\nu} - k^\mu k^\nu}_{M_{\mu\nu}k^\nu = 0}]A_\nu(k)$$

zero eigenvector

$$ik_\mu \to \partial_\nu$$

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

family of zero eigenvalues

$$M_{\mu\nu}\underbrace{f k^{\nu}} = 0$$

generator of gauge symmetry

1.) rotational invariance

2.) A is still a 1-form

3.) $[f, k_\mu] = 0$

only choice

$$f \equiv f(k^2)$$

$$(\Delta)^{\gamma}$$

$$A_\mu \to A_\mu + (\Delta)^{\frac{(\gamma-1)}{2}} \partial_\mu \Lambda$$

$$[A_\mu] = \gamma$$

what kind of E&M has such gauge transformations?

if

$$[A_\mu] \neq 1$$

but the current is conserved

extra dimensions

# model with anomalous dimensions

$$S = \int \mathrm{d}^{d+2}x\sqrt{-g}\left[\mathcal{R} - \frac{(\partial_\mu\phi)^2}{2} - \frac{Z(\phi)}{4}F^2 + V(\phi)\right]$$

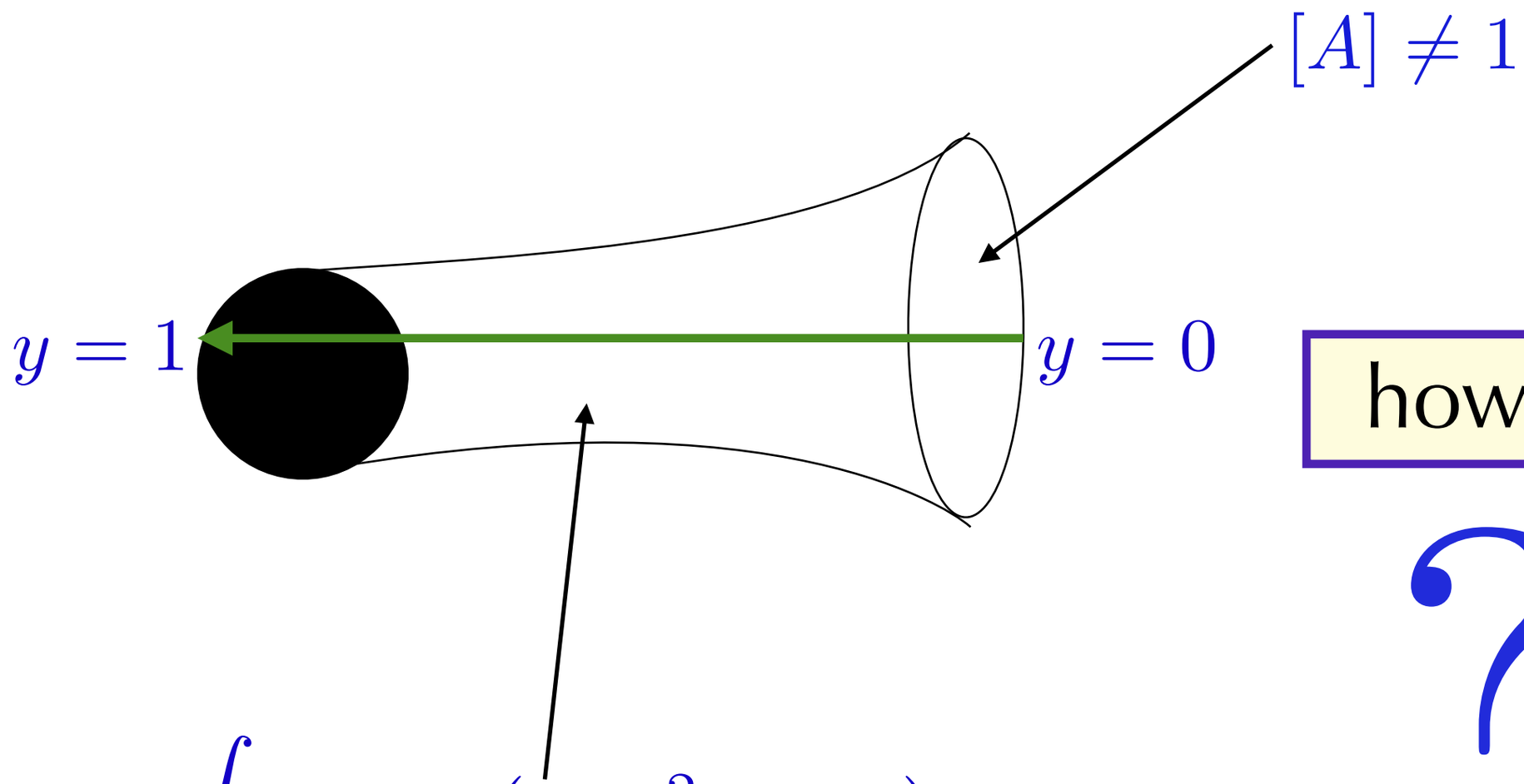
$$\begin{cases} Z(\phi) \underset{\phi\to\infty}{\to} Z_0 e^{\gamma\phi} \\ V(\phi) \underset{\phi\to\infty}{\to} V_0 e^{-\delta\phi}. \end{cases}$$

$$e^{\phi} = r^{\pm\kappa}$$

$$r^{\alpha}F^2$$

Karch:1405.2926
Gouteraux: 1308.2084

claim




how?

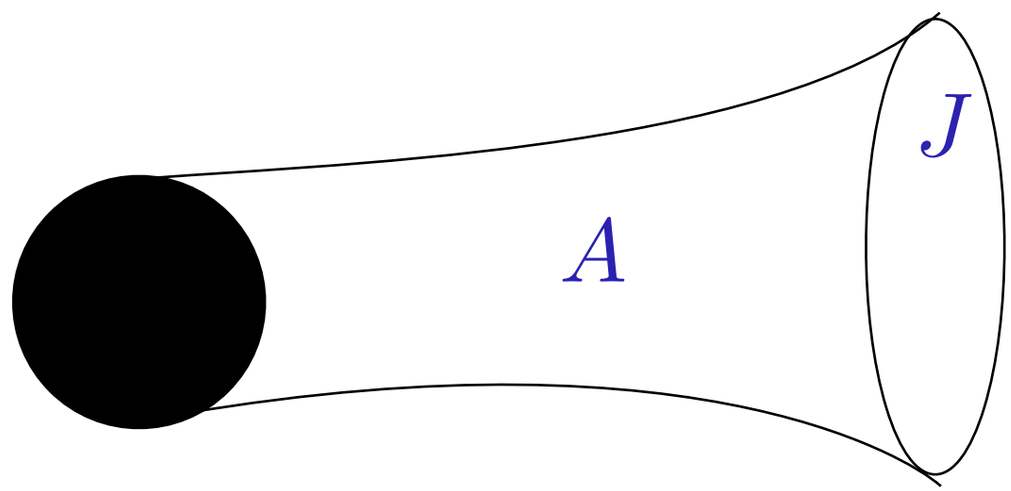
?

$$S = \int dV_d dy \left( y^a F^2 + \cdots \right)$$

$$F = dA$$

if holography is RG then how can it lead to an anomalous dimension?

standard case




$$\int d^d x A_{||} J$$

$$A(y=0) = A_{||}$$

bc does not satisfy

$$A(y=0) \neq A_{||} + d\Lambda$$

alternatively

$$(A + d\Lambda)_{\partial\Omega} = a + d^{||}\Lambda_{\partial\Omega}$$

boundary theory has non-trivial gauge structure

AdS/ Lifshitz

$$\int dy/y = \infty$$

large gauge transformation

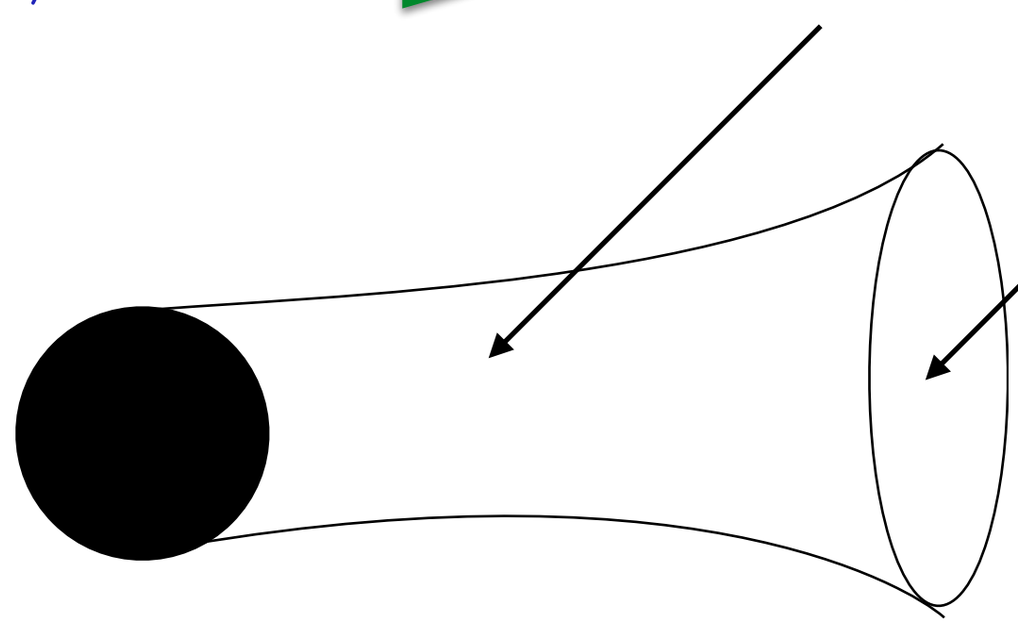
construct boundary theory explicitly

$$S = \int dV_d dy \left( y^a F^2 + \cdots \right)$$

eom

$$d(y^a \star dA) = 0$$

$$y \neq 0 \longrightarrow A \to A + \partial\Lambda$$



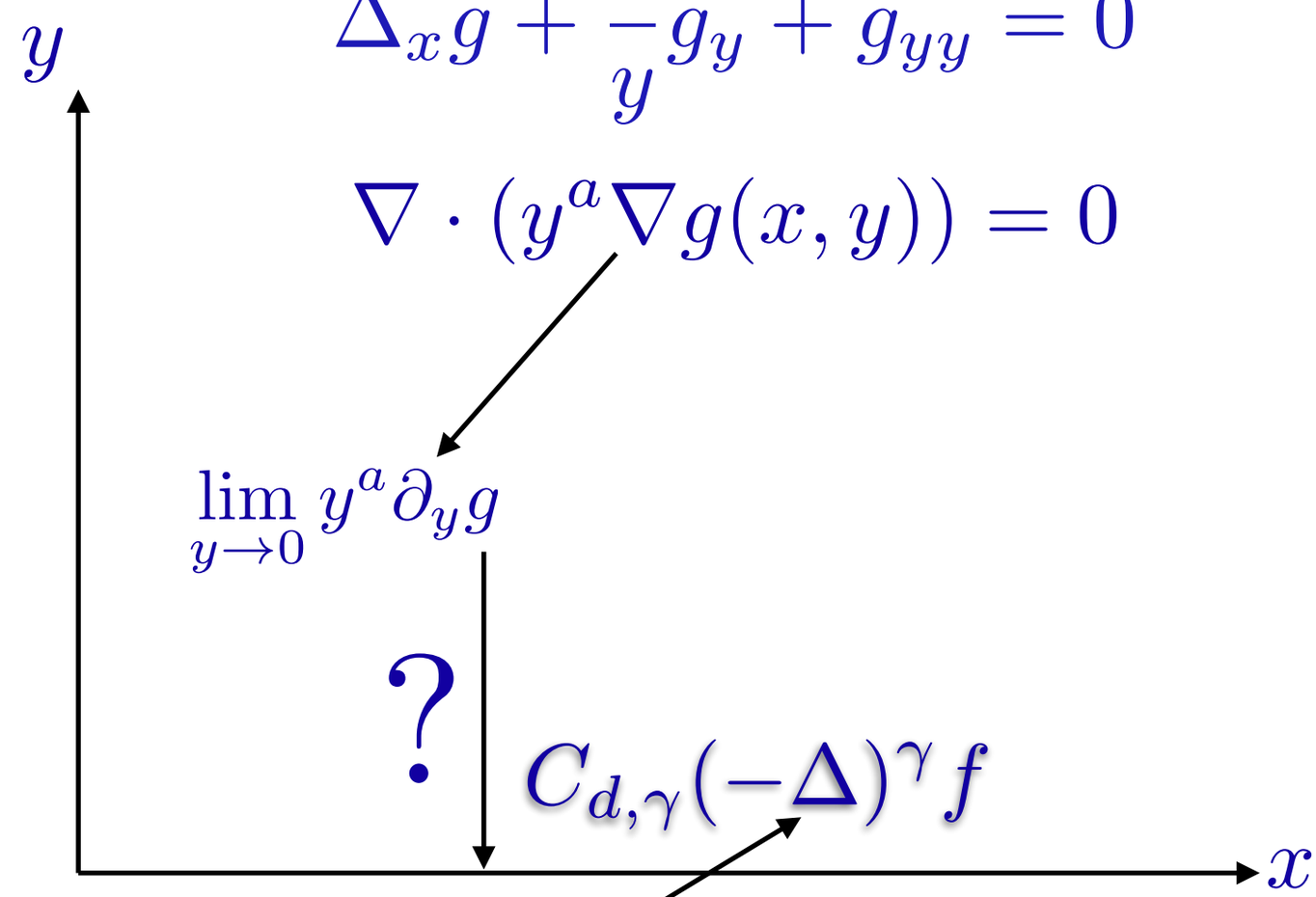
what about the boundary?

Caffarelli-Silvestre extension theorem (2006)

$$g(x, y = 0) = f(x)$$

$$\Delta_x g + \frac{a}{y} g_y + g_{yy} = 0$$

$$\nabla \cdot (y^a \nabla g(x, y)) = 0$$




fractional Laplacian

$$g(z = 0, x) = f(x)$$

$$\gamma = \frac{1 - a}{2}$$

boundary is non-local!

$$(-\Delta_x)^a f(x) = C_{d,a} \int_{R^d} d\xi \frac{(f(x) - f(\xi))}{|x - \xi|^{d+2a}}$$

# closer look

$$\nabla \cdot (y^a \nabla u) = 0$$

scalar field
(use CS theorem)

$$d(y^a \star dA) = 0$$

holography

similar equations

generalize CS
theorem to p-forms
GL,PP:1708.00863

generalize CS to p-forms

$$\lim_{y\to 0} y^a \frac{\partial \alpha_{i_1\cdots i_p}}{\partial y} = C_{d,\gamma}(-\Delta)^\gamma \omega_{i_1\cdots i_p}$$

fractional Maxwell equations at boundary!

boundary action: fractional Maxwell equations

$$\Delta^\gamma A_\perp = 0$$

$$\Delta^\gamma A_\perp = J$$

boundary action has `anomalous dimension' (non-locality)

if holography is RG then how can it lead to an anomalous dimension?

$$S = \int dV_d dy \left( y^a F^2 + \cdots \right)$$

$$[A] = 1 - a/2$$

dimension of A is fixed by the bulk theory: not really anomalous dimension

define

$$F_{ij} = \partial_i^\gamma A_j - \partial_j^\gamma A_i \equiv d_\gamma A = d\Delta^{\frac{\gamma-1}{2}} A,$$

$$S = \int -\frac{1}{4} F_{ij} F^{ij}$$

integrate by parts

$$S = \int \frac{1}{2} A_i (-\Delta)^{2\gamma} A^i,$$

non-local boundary action

# fractional differential

$$d_a = \frac{1}{2}(d(\overbrace{d^* d}^{\Delta^{(a-1)/2}})^{(a-1)/2}\omega + (dd^*)^{(a-1)/2} d\omega)$$

$$d^* = (-1)^{n(p+1)+1} \star d \star$$

$$dd^* : \Omega^p(M) \to \Omega^p(M)$$

does not change the order of the form

new gauge transformation

$$A \to A + d_\gamma \Lambda,$$

$$d_\gamma \equiv (\Delta)^{\frac{\gamma-1}{2}} d$$

$$[A] = \gamma$$

boundary does not lie at infinity
(not a large gauge transformation)

$$S = \int \frac{1}{2} A_i (-\Delta)^{2\gamma} A^i,$$

$$\Downarrow$$

$$\langle A^i(k) A^j(-k) \rangle \propto \frac{\eta^{ij}}{k^{2\gamma}}$$

$$\partial_\mu J^\mu = 0$$ current conservation

what if

$$[\partial_\mu, \hat{Y}] = 0$$

answer

?

$$[\partial_\mu, \hat{Y}] = 0$$

$$[d, \Delta^\alpha] = 0$$

$$\hat{Y} = \Delta^\alpha$$

$$J \to \Delta^\alpha J \qquad [J] = d - 1 - \alpha$$

current-current correlator

$$C^{ij}(k) \propto (k^2)^\gamma \left( \eta^{ij} - \frac{k^i k^j}{k^2} \right).$$

standard Ward identity

$$k_i C^{ij}(k) = 0 \Rightarrow \partial_i C^{ij}(k) = 0$$

but

$$k^{\gamma-1} k_\mu C^{\mu\nu} = 0 \Rightarrow \partial_\mu (-\Delta)^{\frac{\gamma-1}{2}} C^{\mu\nu} = 0$$

inherent ambiguity in E&M

family of zero eigenvalues

$$M_{\mu\nu} f k^{\nu} = 0$$

# Noether's Theorems

$$\sum \psi_i \delta u_i = \delta f -$$

$$- \frac{d}{dx}\left\{\sum\left[\binom{1}{1}\frac{\partial f}{\partial u_i^{(1)}}\delta u_i + \binom{2}{1}\frac{\partial f}{\partial u_i^{(2)}}\delta u_i^{(1)} + \cdots + \binom{\kappa}{1}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_i^{(\kappa-1)}\right]\right\} +$$

$$+ \frac{d^2}{dx^2}\left\{\sum\left[\binom{2}{2}\frac{\partial f}{\partial u_i^{(2)}}\delta u_i + \binom{3}{2}\frac{\partial f}{\partial u_i^{(3)}}\delta u_i^{(1)} + \cdots + \binom{\kappa}{2}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_i^{(\kappa-2)}\right]\right\} +$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$+ (-1)^\kappa \frac{d^\kappa}{dx^\kappa}\left\{\sum\binom{\kappa}{\kappa}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_1\right\} \tag{6}$$

$$A_\mu \to A_\mu + \partial\Lambda + f_1\partial^2\Lambda + \cdots$$

# Noether's Second Theorem and Ward Identities for Gauge Symmetries

Steven G. Avery[a], Burkhard U. W. Schwab[b]

For simplicity, we focus on the case when the transformation may be written in the form[6]

$$\delta_\lambda \phi = f(\phi)\lambda + f^\mu(\phi)\partial_\mu \lambda, \tag{10}$$

but it is straightforward to consider transformations, as Noether did, involving arbitrarily high derivatives of $\lambda$. (Although, the authors know of no physically interesting examples.) Let us start with

arxiv:1510.07038

is this just a game?

is there a consistent algebra for fractional currents?

Yes

# Virasoro algebra

$$L_n := -z^{n+1}\frac{\partial}{\partial z}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

Witt algebra

central extension

conformal transformations on unit disk

$$\mathcal{V} \to \mathcal{W} \to 1$$

# Fractional Virasoro algebra

## generators

$$L_n^a = -z^{a(n+1)}\left(\frac{\partial}{\partial z}\right)^a \qquad \bar{L}_n^a := -\bar{z}^{a(n+1)}\left(\frac{\partial}{\partial \bar{z}}\right)^a$$

$$[L_n, L_m](z^{ak}) = \left(\frac{\Gamma(a(k+n)+1)}{\Gamma(a(k-1+n)+1)} - \frac{\Gamma(a(k+m)+1)}{\Gamma(a(k-1+m)+1)}\right) L_{n+m}(z^{ak})$$

$$= (A^a_{n,m}(k) \otimes L_{n+m})(z^{ak})$$

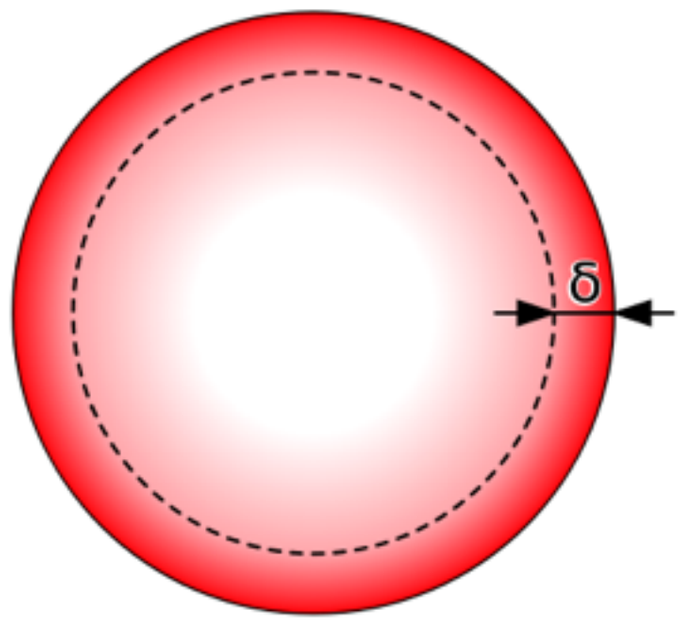
$$\boxed{[L_m^a, L_n^a] = A_{m,n} L^a_{m+n} + \delta_{m,n} h(n) c Z^a}$$

algebra for conformal non-local actions

$$Z^2_\star(\mathcal{W}_a, \mathcal{H})/B^2_\star(\mathcal{W}_a, \mathcal{H})$$

experiments?

# skin effect

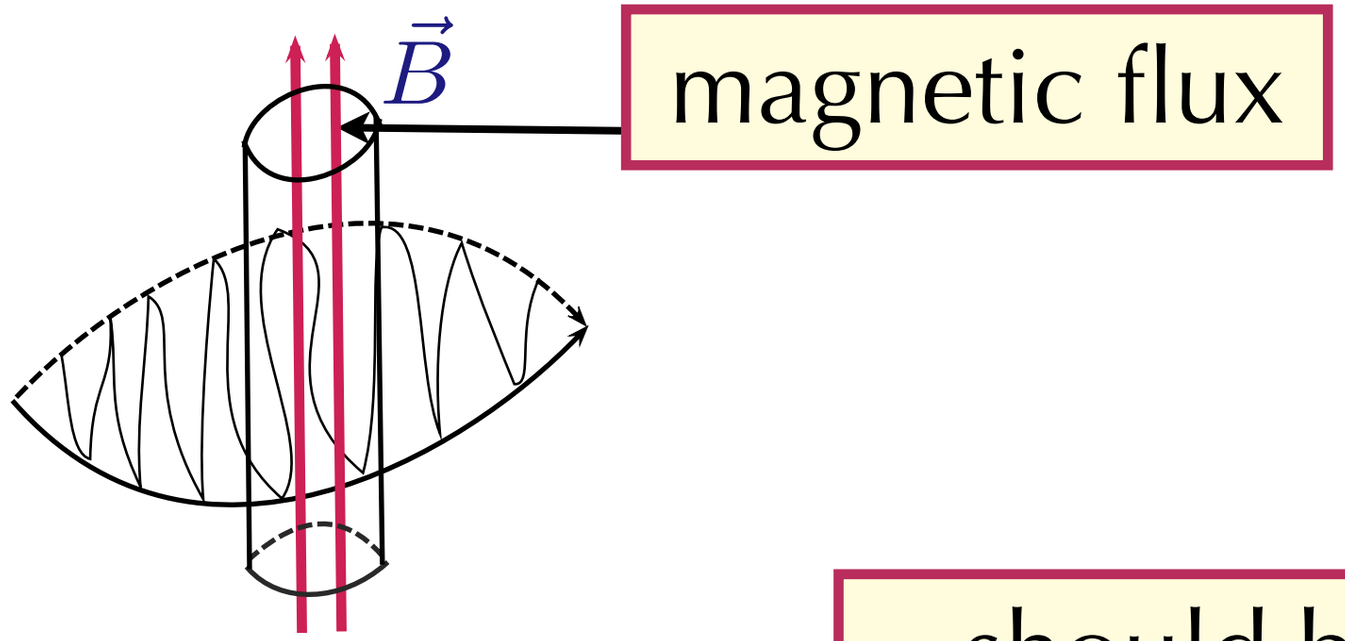



$$\delta = \sqrt{\frac{2\rho}{\omega\mu}}$$

new result

$$\Box^{\frac{\gamma-1}{2}}\left(\nabla\times\vec{B}-\frac{1}{v^2}\frac{\partial\vec{E}}{\partial t}\right)=\mu\vec{J}$$

$$\delta=1/k_2=\left(\frac{\epsilon v^2}{\omega\sigma}\right)^{\frac{1}{2(\gamma+1)}}\frac{1}{\sin\left(\frac{\pi}{2(\gamma+1)}+\frac{2\pi n}{\gamma+1}\right)}$$




$\pi r^2 B$

should be dimensionless

$$[B] = 2 - \Phi = 2 + 2/3 \neq 2$$

what's the resolution?

correct dimensionless quantity

$$D_i \equiv \partial_i - i\frac{e}{\hbar}a_i$$

fictitious gauge field

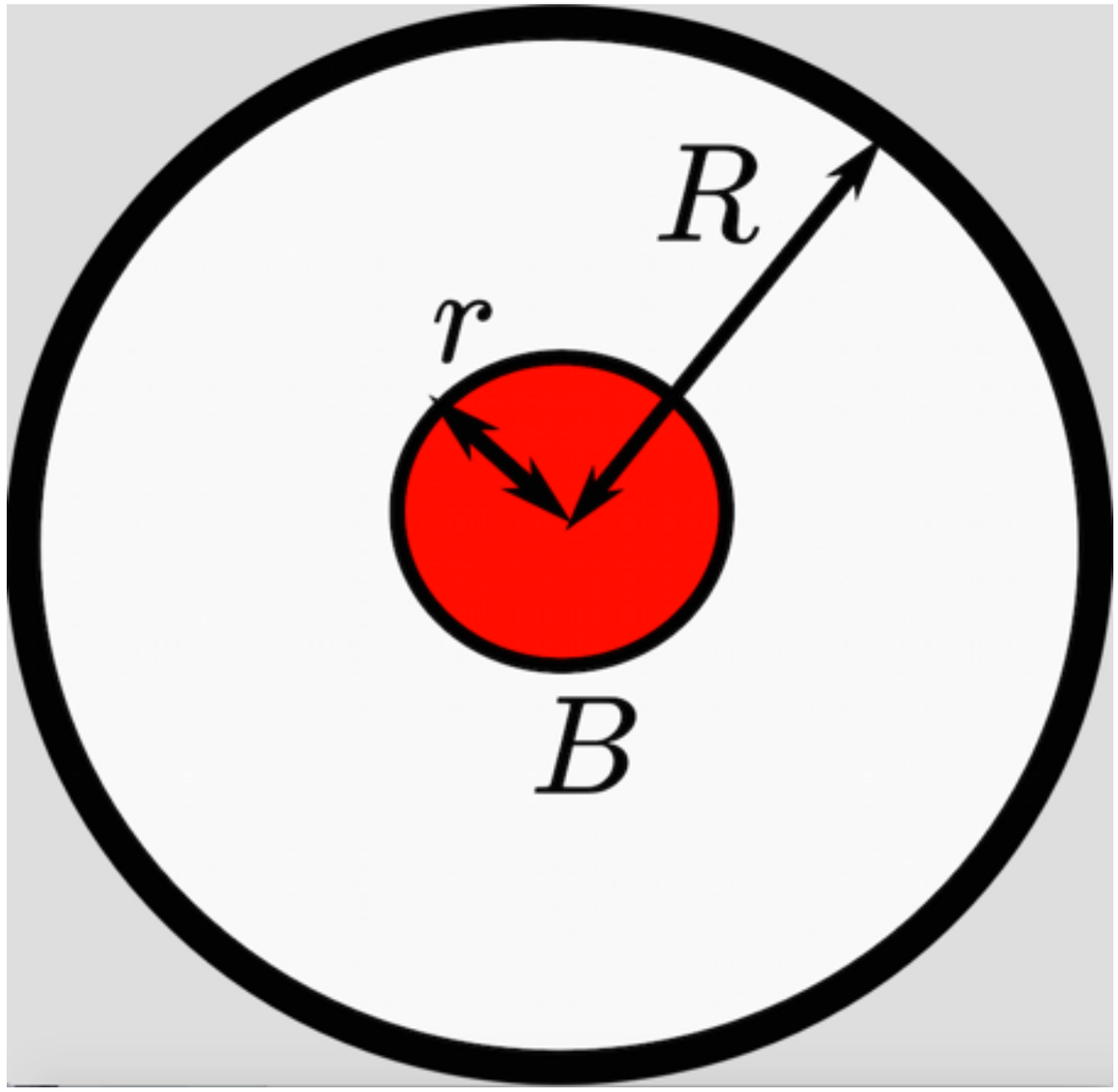
$$a_i \equiv [\partial_i, I_i^{\alpha}{}_{\alpha}A_i] = \partial_i I_i^{\alpha}{}_{\alpha}A_i$$

$${}_{\alpha}A \rightarrow_{\alpha} A + d_{\alpha}\Lambda$$

$$a_\mu \rightarrow a_\mu + \partial_\mu \Lambda$$

$$\Delta\phi = \frac{e}{\hbar}\oint \vec{a}(\vec{r}) \cdot d\vec{l}.$$

# New Aharonov-Bohm Effect




$$\Delta\phi_{\mathrm{D}} = \frac{e}{\hbar}\pi r^2 B R^{2\alpha-2}\left(\frac{\sqrt{\pi}2^{1-\alpha}\Gamma(2-\alpha)\Gamma(1-\frac{\alpha}{2})}{\Gamma(\alpha)\Gamma(\frac{3}{2}-\frac{\alpha}{2})}\sin^2\frac{\pi\alpha}{2}\,{}_2F_1(1-\alpha,2-\alpha;2;\frac{r^2}{R^2})\right)$$

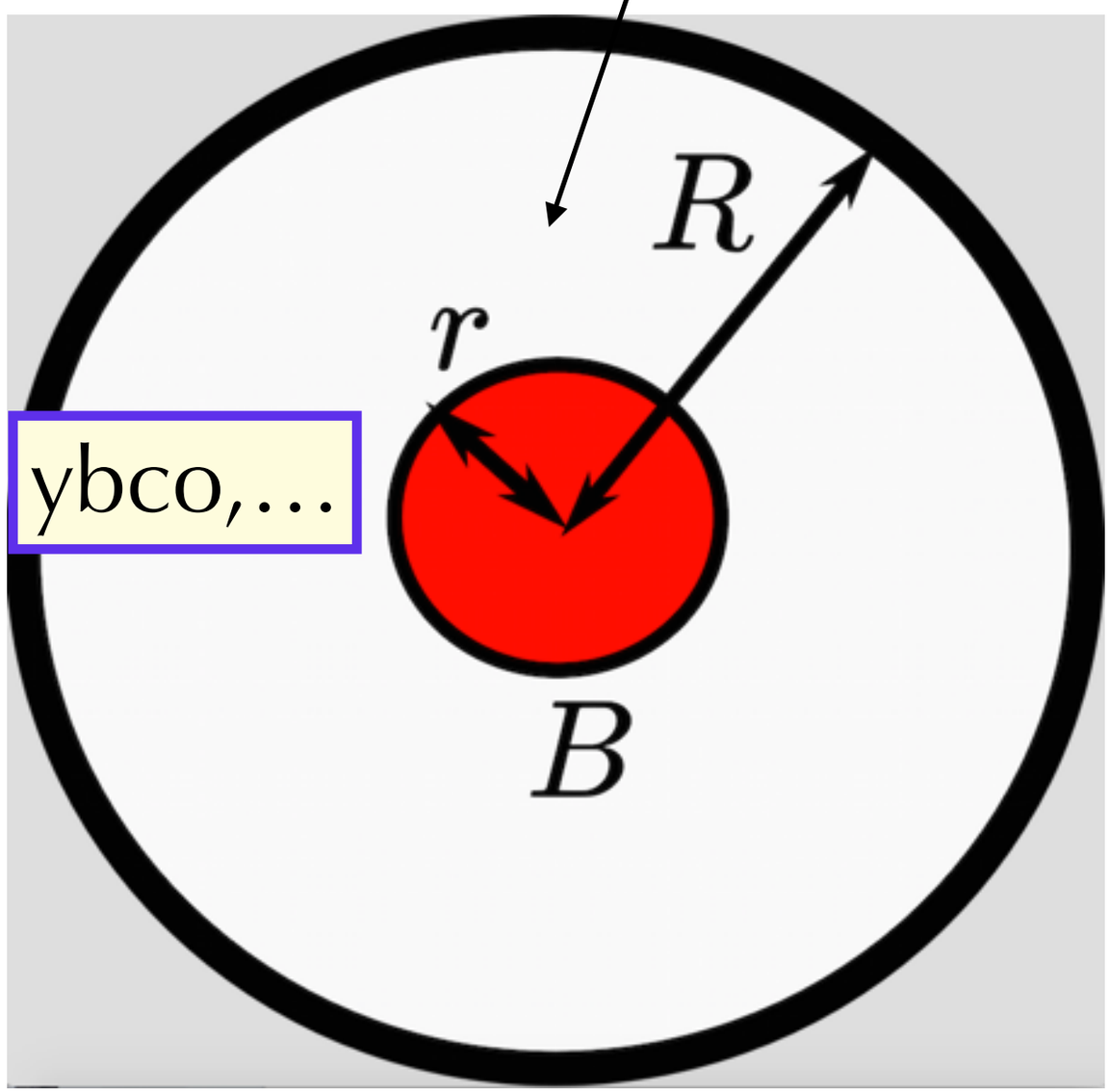
is the correction large?

$$\alpha = 1 + 2/3 = 5/3$$

$$\Delta\Phi_R = \frac{eB\ell^2}{\hbar} L^{-5/3}/(0.43)^2$$

yes!

experiment




Planckian dissipation

$$\tau = \frac{\hbar}{k_B T}$$

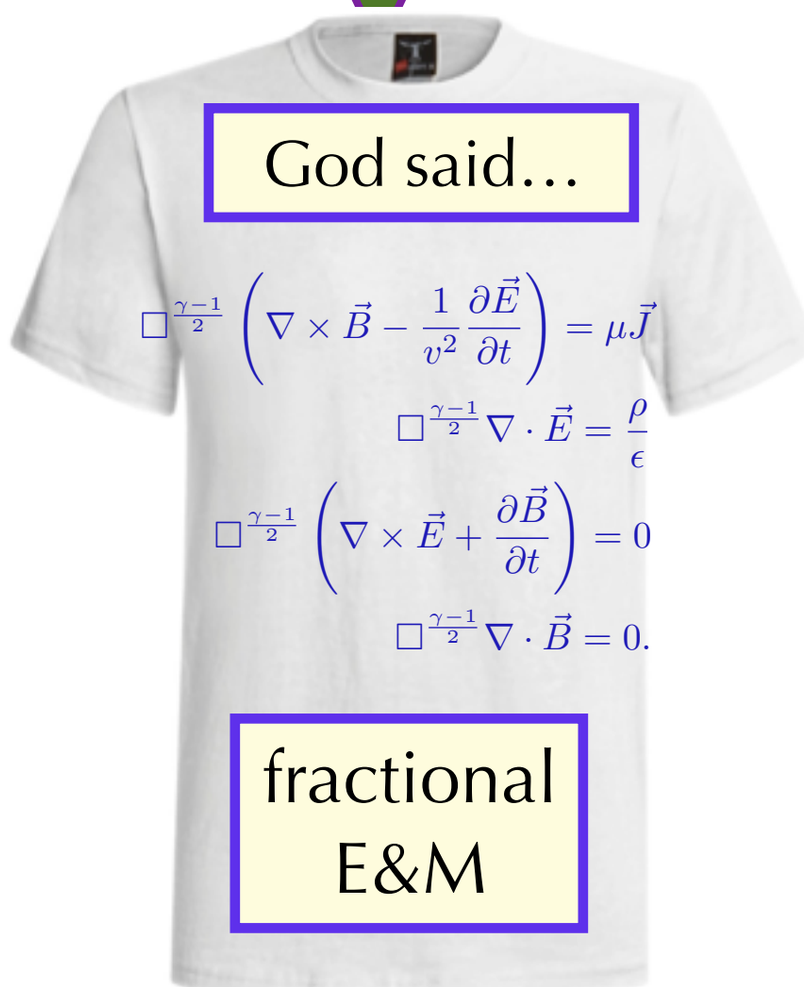
$$\tau \approx 10^{-14} s$$

**Table 11.1**

| Element | 77 K | 273 K |
|---|---|---|
| Li | $7.3 \times 10^{-14} s$ | $8.8 \times 10^{-15} s$ |
| Na | $1.7 \times 10^{-13} s$ | $3.2 \times 10^{-14} s$ |
| K | $1.8 \times 10^{-13} s$ | $4.1 \times 10^{-14} s$ |
| Rb | $1.4 \times 10^{-13} s$ | $2.8 \times 10^{-14} s$ |
| Cs | $8.6 \times 10^{-14} s$ | $2.1 \times 10^{-14} s$ |

if in the strange metal

$$[A_\mu] = d_A \neq 1$$

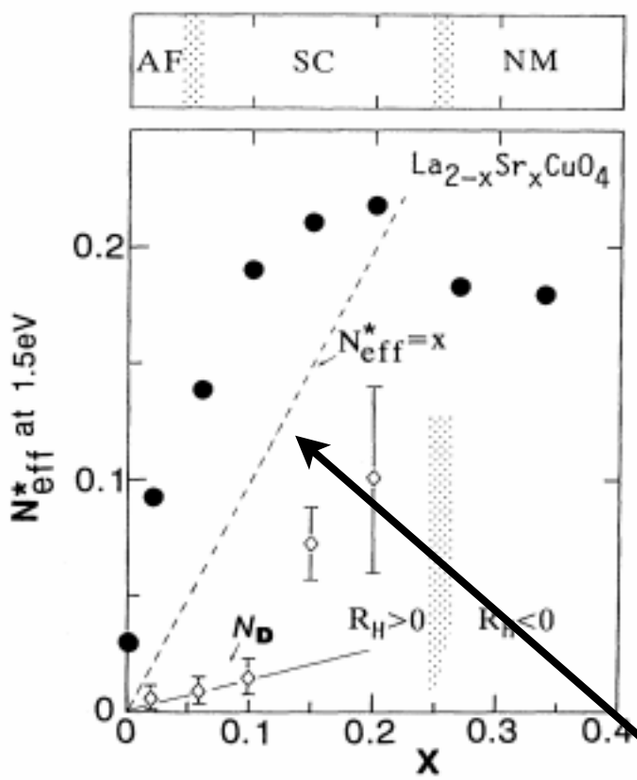


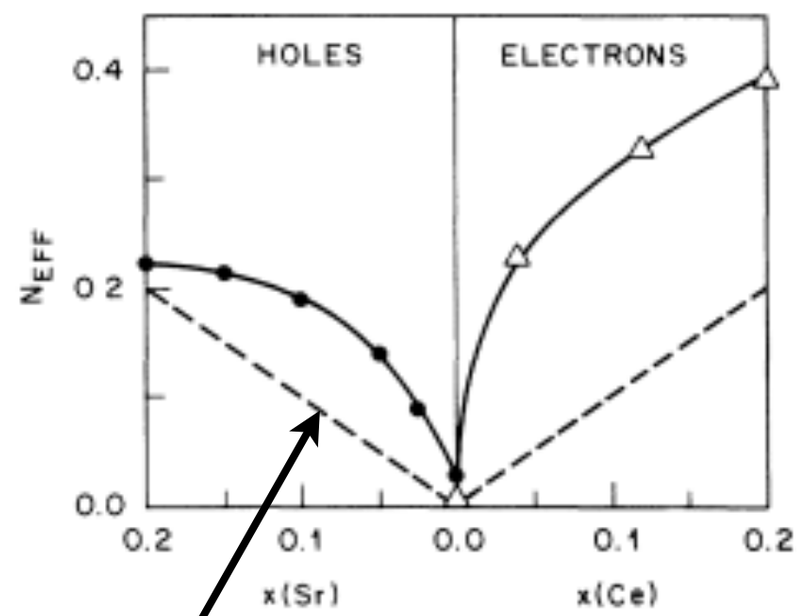
what else can be explained
with the anomalous dimension?

$$N_{\text{eff}}(\Omega) = \frac{2mV_{\text{cell}}}{\pi e^2}\int_0^{\Omega}\sigma(\omega)d\omega$$

Uchida, et al.

Cooper, et al.




$x$

low-energy model for $N_{\text{eff}} > x$??

f-sum rule

$$\text{K.E.} = p^2/2m$$

$$N_{\text{eff}} = x$$

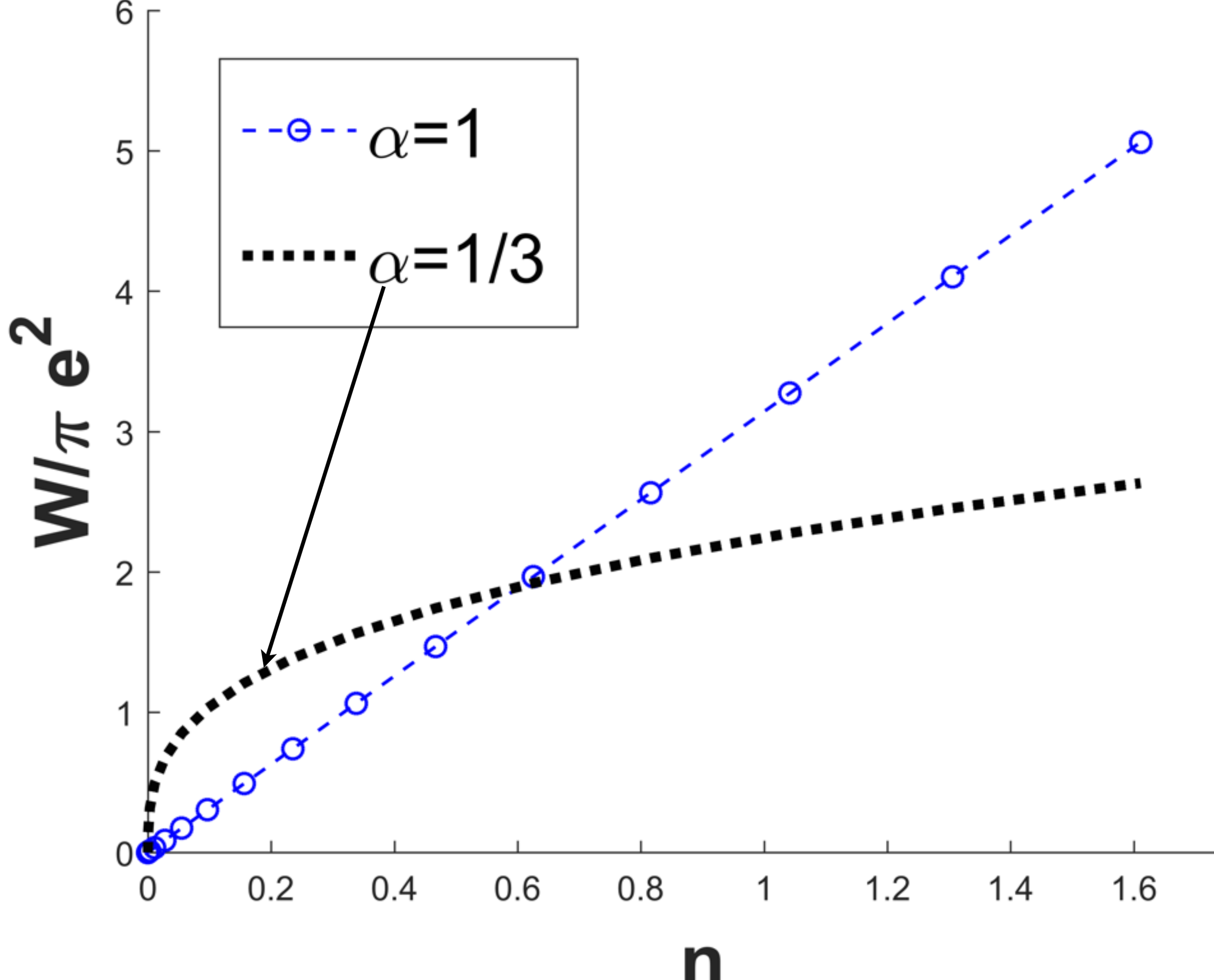
what if?

$$\text{K.E.} \propto (\partial_\mu^2)^\alpha$$

f-sum rule

$$\frac{W(n,T)}{\pi c e^2} = A n^{1+\frac{2(\alpha-1)}{d}} + \cdots$$

W>n if $\alpha < 1$


α=1
α=1/3
W/π e²
n
0
1
2
3
4
5
6
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6