`Anomalous dimensions' for conserved currents and Noether’s Second Theorem

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Mott insulator

no order

0
Strange metal: experimental facts

\[ L_{xy} = \frac{\kappa_{xy}}{T \sigma_{xy}} \neq \# \propto T \]

\[ \sigma(\omega) = C \omega^{-\frac{2}{3}} \]

\[ \frac{n \tau e^2}{m} \frac{1}{1 - i \omega \tau} \]
standard metals

resistivity

\[ \rho \propto T^2 \]

Weidemann-Franz law

\[ \frac{\kappa_{xx}}{T\sigma_{xx}} = \frac{\pi^2}{3} \]

optical conductivity

\[ \Re \sigma \propto \frac{1}{\omega^2} \]
why is the problem hard?
single-parameter scaling

\[ \rho \propto T \]

\[ \sigma(\omega, T) \propto \omega \]

\[ \mathcal{C} \propto T^{d/z} \]

\[ \frac{\delta^2 B}{\omega \delta A_\mu \delta A} \]

\[ -2/3 \]

anomalous dimension

\[ 2d_A \]
The dimension of the vector potential $A_\mu$ is $1/L = 1$.

Gauge invariance

$$A_\mu \rightarrow A_\mu + \partial_\mu G$$

has no units

Vector potential cannot have an anomalous dimension

Conserved current
gauge invariance = current conservation

\[ S = \int d^d x \left( F^2 + J_\mu A^\mu + \cdots \right) \]

\[ A \to A + \partial \Lambda \]

\[ S \to S + \int d^d x J_\mu \partial \Lambda \]

integrate by parts

\[ \partial_\mu J^\mu = 0 \]

Noether’s Thm. I

current conservation
gauge invariance

$[A_\mu] = 1$

current conservation

$[d^d x J A] = 0$

fixes dimension of current

$[J] = d - 1$
general argument

\[ \phi(x) \rightarrow \phi(x) + \delta \phi(x) \]

\[ [J_0(x), \phi(y)] = \delta \phi(x) \delta^d(x - y) \]

\[ [J_0] = d \]
\[ \langle A^i(k) A^j(-k) \rangle \propto \frac{\eta^{ij}}{k^2} \]
strange metal explained!

**Hall Angle**

\[ \cot \theta_H \equiv \frac{\sigma_{xx}}{\sigma_{xy}} \approx T^2 \]

**Hall Lorenz ratio**

\[ L_{xy} = \frac{\kappa_{xy}}{T \sigma_{xy}} \neq \# \propto T \]

all explained if

\[ [J_\mu] = d - \theta + \Phi + z - 1 \]

Hartnoll/Karch

\[ [A_\mu] = 1 - \Phi \]

\[ \Phi = -2/3 \]
\[ [J_\mu] = d - \theta + \Phi + z - 1 \]
\[ [A_\mu] = 1 - \Phi \]
\[ \Phi = -2/3 \]
\[ [E] = 1 + z - \Phi \]
\[ [B] = 2 - \Phi \]

\[ \langle A^i(k)A^j(-k) \rangle \propto \frac{\eta^{ij}}{k^2(1-\Phi)} \]

\text{note} \quad \pi r^2 B \neq \text{flux}

\text{non-local propagator}
Scaling theory of conserved current and universal amplitudes at anisotropic critical points

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A scaling theory of conserved currents is studied at critical points with critical charge fluctuations. It is shown that the conserved current has no anomalous dimension even for anisotropic critical points. Many universal amplitudes are identified. One of the universal amplitudes at the critical points is

$$\lim_{(k, \omega) \to 0} \omega^{2-d} k^{(z-d)(2-d)} [\sigma(\omega)]^{2} [\kappa(k)]^{2-d},$$

which reduces to the universal conductance in 2+1 dimensions. We find that the ratio $\sqrt{\kappa \chi}/\sigma$ always approaches a constant as we approach the critical point. We also determine exponents of many scaling relations, using the properties of the current.
How is this possible - - if at all?
current algebra with an anomalous dimension?

is there a direct experimental probe of this anomaly?

how can A have an anomalous dimension?

what else can be explained with an anomalous dimension?
what is the new gauge principle?

\[ [A_\mu] \neq 1 \]
hint

\[ \partial_\mu J^\mu = 0 \]  
**current conservation**

what if

\[ [\partial_\mu, \hat{Y}] = 0 \]

**new current**

\[ [\tilde{J}] = d - 1 - D_Y \]

\[ \partial_\mu \hat{Y} J^\mu = \partial_\mu \tilde{J}^\mu = 0 \]
possible gauge transformations

\[ S = -\frac{1}{4} \int d^d x F^2 \]

\[ S = \frac{1}{2} \int \frac{d^d k}{2\pi^d} A_\mu(k) [k^2 \eta^{\mu\nu} - k^\mu k^\nu] A_\nu(k) \]

\[ M_{\mu\nu} k^\nu = 0 \]

zero eigenvector

\[ ik_\mu \rightarrow \partial_\nu \]

\[ A_\mu \rightarrow A_\mu + \partial_\mu \Lambda \]
family of zero eigenvalues

\[ M_{\mu \nu} f k^\nu = 0 \]

generator of gauge symmetry

1.) rotational invariance
2.) A is still a 1-form
3.) \([f, k_\mu] = 0\)
only choice

\[ f = f(k^2) \]

\[ (\Delta)\gamma \]

\[ A_\mu \rightarrow A_\mu + (\Delta) \frac{(\gamma-1)}{2} \partial_\mu \Lambda \]

\[ [A_\mu] = \gamma \]

what kind of E&M has such gauge transformations?
if

\[ [A_\mu] \neq 1 \]

but the current is conserved
extra dimensions
model with anomalous dimensions

\[
S = \int d^{d+2}x \sqrt{-g} \left[ \mathcal{R} - \frac{(\partial_{\mu}\phi)^2}{2} - \frac{Z(\phi)}{4} F^2 + V(\phi) \right]
\]

\[
\begin{align*}
Z(\phi) &\to Z_0 e^{\gamma \phi} \\
V(\phi) &\to V_0 e^{-\delta \phi}.
\end{align*}
\]

\[e^{\phi} = r^{\pm \kappa}\]

\[r^\alpha F^2\]

Karch:1405.2926
Gouteraux: 1308.2084
\[ S = \int dV_{\alpha}dy \left( y^{\alpha} F^2 + \cdots \right) \]

\[ F = dA \]
if holography is RG then how can it lead to an anomalous dimension?
standard case

\[ A(y = 0) = A_{\parallel} + d\Lambda \]

bc does not satisfy
alternatively

\[(A + d\Lambda)_{\partial\Omega} = a + d^\parallel \Lambda_{\partial\Omega}\]

boundary theory has non-trivial gauge structure

AdS/Lifshitz

\[\int dy/y = \infty\]

large gauge transformation
construct boundary
theory explicitly
\[ S = \int dV_d dy \left( y^a F^2 + \cdots \right) \]

\[ d(y^a \ast dA) = 0 \]

\[ y \neq 0 \quad \rightarrow \quad A \rightarrow A + \partial \Lambda \]

what about the boundary?
\[ g(x, y = 0) = f(x) \]
\[ \Delta_x g + \frac{a}{y} g_y + g_{yy} = 0 \]
\[ \nabla \cdot (y^a \nabla g(x, y)) = 0 \]
\[ \lim_{y \to 0} y^a \partial_y g \]
\[ C_{d, \gamma} (-\Delta)^\gamma f \]
\[ g(z = 0, x) = f(x) \]
\[ \gamma = \frac{1 - a}{2} \]
boundary is non-local!

\[ (-\Delta_x)^a f(x) = C_{d,a} \int_{\mathbb{R}^d} d\xi \frac{(f(x) - f(\xi))}{|x - \xi|^{d+2a}} \]
closer look

\[ \nabla \cdot (y^a \nabla u) = 0 \]

scalar field (use CS theorem)

\[ d(y^a \star dA) = 0 \]

holography

similar equations

generalize CS theorem to p-forms

GL,PP:1708.00863
generalize CS to p-forms

$$\lim_{y \to 0} y^a \frac{\partial \alpha_{i_1 \ldots i_p}}{\partial y} = C_{d,\gamma} (-\Delta)^{\gamma} \omega_{i_1 \ldots i_p}$$

fractional Maxwell equations at boundary!
boundary action: fractional Maxwell equations

$$\Delta^\gamma A_\perp = 0$$

$$\Delta^\gamma A_\perp = J$$

boundary action has `anomalous dimension' (non-locality)
if holography is RG then how can it lead to an anomalous dimension?

\[ S = \int dV d\sigma \left( y^a F^2 + \cdots \right) \]

\[ [A] = 1 - a/2 \]

dimension of A is fixed by the bulk theory: not really anomalous dimension
define

\[ F_{ij} = \partial_i^\gamma A_j - \partial_j^\gamma A_i \equiv d\gamma A = d\Delta^{\frac{\gamma-1}{2}} A, \]

\[ S = \int -\frac{1}{4} F_{ij} F^{ij} \]

\[ S = \int \frac{1}{2} A_i (-\Delta)^{2\gamma} A^i, \]

integrate by parts

non-local boundary action
Fractional differential

\[ d_a = \frac{1}{2} \left( d(d^*d)^{(a-1)/2} \omega + (dd^*)^{(a-1)/2} d\omega \right) \]

\[ d^* = (-1)^{n(p+1)+1} * d^* \]

\[ dd^* : \Omega^p(M) \to \Omega^p(M) \]

does not change the order of the form
new gauge transformation

\[ A \rightarrow A + d_\gamma \Lambda, \]

\[ d_\gamma \equiv (\Delta)^{\frac{\gamma-1}{2}} d \]

\[ [A] = \gamma \]

boundary does not lie at infinity
(not a large gauge transformation)
\[ S = \int \frac{1}{2} A_i (-\Delta)^{2\gamma} A^i, \]

\[ \langle A^i (k) A^j (-k) \rangle \propto \frac{\eta^{ij}}{k^{2\gamma}} \]
$\partial_{\mu} J^{\mu} = 0$  

current conservation

what if

$[\partial_{\mu}, \hat{Y}] = 0$
\[ [\partial_\mu, \hat{Y}] = 0 \]
\[ [d, \Delta^\alpha] = 0 \]
\[ \hat{Y} = \Delta^\alpha \]
\[ J \rightarrow \Delta^\alpha J \quad [J] = d - 1 - \alpha \]
current-current correlator

\[ C^{ij}(k) \propto (k^2)^\gamma \left( \eta^{ij} - \frac{k^i k^j}{k^2} \right). \]

standard Ward identity

\[ k_i C^{ij}(k) = 0 \quad \Rightarrow \quad \partial_i C^{ij}(k) = 0 \]

but

\[ k^{\gamma -1} k_\mu C^{\mu \nu} = 0 \quad \Rightarrow \quad \partial_\mu (-\Delta)^{\frac{\gamma - 1}{2}} C^{\mu \nu} = 0 \]

inherent ambiguity in E&M
family of zero eigenvalues

\[ M_{\mu\nu} f k^\nu = 0 \]
Noether’s Theorems

\[ \sum_{\mu} \delta u_{\mu} = \delta f - \]

\[- \frac{d}{dx} \left\{ \sum \left[ \frac{\partial f}{\partial u_1} \delta u_1 + \frac{\partial f}{\partial u_2} \delta u_2 + \cdots + \frac{\partial f}{\partial u_\kappa} \delta u_\kappa \right] \right\} + \]

\[+ \frac{d^2}{dx^2} \left\{ \sum \left[ \frac{\partial^2 f}{\partial u_1} \delta u_1 + \frac{\partial^2 f}{\partial u_2} \delta u_2 + \cdots + \frac{\partial^2 f}{\partial u_\kappa} \delta u_\kappa \right] \right\} + \]

\[+ \cdots \]

\[+ (-1)^\kappa \frac{d^\kappa}{dx^\kappa} \left\{ \sum \left[ \frac{\partial^\kappa f}{\partial u_1} \delta u_1 \right] \right\} \]

(6)

\[ A_\mu \rightarrow A_\mu + \partial \Lambda + f_1 \partial^2 \Lambda + \cdots \]
Noether’s Second Theorem and Ward Identities for Gauge Symmetries

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For simplicity, we focus on the case when the transformation may be written in the form\textsuperscript{6}

\[
\delta_{\lambda} \phi = f(\phi) \lambda + f^{\mu}(\phi) \partial_\mu \lambda,
\]

but it is straightforward to consider transformations, as Noether did, involving arbitrarily high derivatives of $\lambda$. (Although, the authors know of no physically interesting examples.) Let us start with

\[\text{arxiv:1510.07038}\]
is this just a game?

is there a consistent algebra for fractional currents?
Yes
Virasoro algebra

\[ L_n := -z^{n+1} \frac{\partial}{\partial z} \]

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} \]

Witt algebra

central extension

conformal transformations on unit disk

\[ \mathcal{V} \rightarrow \mathcal{W} \rightarrow 1 \]
Fractional Virasoro algebra

generators

\[ L_n^a = -z^{a(n+1)} \left( \frac{\partial}{\partial z} \right)^a \quad \bar{L}_n^a := -\bar{z}^{a(n+1)} \left( \frac{\partial}{\partial \bar{z}} \right)^a \]

\[
[L_n, L_m](z^{ak}) = \left( \frac{\Gamma(a(k+n)+1)}{\Gamma(a(k-1+n)+1)} - \frac{\Gamma(a(k+m)+1)}{\Gamma(a(k-1+m)+1)} \right) L_{n+m}(z^{ak})
\]

\[ = (A_{n,m}^a(k) \otimes L_{n+m})(z^{ak}) \]

\[ [L_m^a, L_n^a] = A_{m,n}^a L_{m+n}^a + \delta_{m,n} h(n)cZ^a \]

algebra for conformal non-local actions

\[ Z^2_*(\mathcal{W}_a, \mathcal{H})/B^2_*(\mathcal{W}_a, \mathcal{H}) \]
experiments?
skin effect

\[ \delta = \sqrt{\frac{2\rho}{\omega \mu}} \]
new result

\[ \Box^{\gamma-1\over 2} \left( \nabla \times \vec{B} - \frac{1}{v^2} \frac{\partial \vec{E}}{\partial t} \right) = \mu \vec{J} \]

\[ \delta = 1/k_2 = \left( \frac{\varepsilon v^2}{\omega \sigma} \right) \frac{1}{2(\gamma+1)} \frac{1}{\sin \left( \frac{\pi}{2(\gamma+1)} + \frac{2\pi n}{\gamma+1} \right)} \]
magnetic flux

$\pi r^2 B$

should be dimensionless

$[B] = 2 - \Phi = 2 + \frac{2}{3} \neq 2$

what’s the resolution?
correct dimensionless quantity

\[ D_i \equiv \partial_i - i \frac{e}{\hbar} a_i \]

fictitious gauge field

\[ a_i \equiv [\partial_i, I_i^\alpha A_i] = \partial_i I_i^\alpha A_i \]

\[ \alpha A \rightarrow_\alpha A + d_\alpha \Lambda \]

\[ a_\mu \rightarrow a_\mu + \partial_\mu \Lambda \]

\[ \Delta \phi = \frac{e}{\hbar} \int \bar{a}(\vec{r}) \cdot d\vec{l}. \]
\[ \Delta \phi_D = \frac{e}{\hbar} \pi r^2 B R^{2\alpha - 2} \left( \frac{\sqrt{\pi} 2^{1-\alpha} \Gamma(2 - \alpha) \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\alpha) \Gamma(\frac{3}{2} - \frac{\alpha}{2})} \sin^2 \frac{\pi \alpha}{2} _2 F_1 (1 - \alpha, 2 - \alpha; 2; \frac{r^2}{R^2}) \right) \]
is the correction large?

$$\alpha = 1 + \frac{2}{3} = \frac{5}{3}$$

$$\Delta \Phi_R = \frac{e B \ell^2}{\hbar} \frac{L^{-5/3}}{(0.43)^2}$$

yes!
Planckian dissipation

\[ \tau = \frac{\hbar}{k_B T} \]

\[ \tau \approx 10^{-14} s \]

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<thead>
<tr>
<th>Table 11.1</th>
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<tbody>
<tr>
<td>Element</td>
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<td>Li</td>
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<td>K</td>
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<td>Rb</td>
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<td>Cs</td>
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</table>
if in the strange metal

\([A_\mu] = d_A \neq 1\)

God said...

\(\Box^{\frac{3}{2}} \left( \nabla \times \vec{B} - \frac{1}{v^2} \frac{\partial \vec{E}}{\partial t} \right) = \mu \vec{J}\)

\(\Box^{\frac{3}{2}} \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}\)

\(\Box^{\frac{3}{2}} \left( \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) = 0\)

\(\Box^{\frac{3}{2}} \nabla \cdot \vec{B} = 0\)

fractional E&M
what else can be explained with the anomalous dimension?
Uchida, et al.

\[ N_{\text{eff}}(\Omega) = \frac{2mV_{\text{cell}}}{\pi e^2} \int_0^\Omega \sigma(\omega) d\omega \]

Cooper, et al.

low-energy model for \( N_{\text{eff}} > x \)
f-sum rule

\[ K.E. = \frac{p^2}{2m} \]

\[ N_{\text{eff}} = x \]
what if?

K.E. $\propto \left( \partial_\mu^2 \right)^\alpha$

f-sum rule

\[
\frac{W(n, T)}{\pi c e^2} = An \left[ 1 + \frac{2(\alpha - 1)}{d} \right] + \cdots
\]

$W > n$ if $\alpha < 1$