Superconductivity and Mottness: Exact Results

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Cooper instability

\[ \langle \psi^\dagger \psi^\dagger \rangle = \Delta e^{i\phi} \]

\[ E_b \propto e^{1/g} \]

\[ -g b_p^\dagger b_{p'} \]

Kammerlingh Onnes (Leiden)
- Liquifies helium 1908
- Discovers superconductivity in mercury 1911
- Receives Nobel Prize 1913

Figure 1: Resistance of a specimen of mercury versus absolute temperature. This plot by Kammerlingh Onnes marked the discovery of superconductivity.
FL $\to$ BCS

Fermi Surface

$S_0 \to S_{\text{repul}}$

1-1 correspondence

$S_{\text{repul}}$ irrelevant

$+S_{\text{pair}}$

superconductivity $\frac{2\Delta}{T_c} = 3.5$

$\beta(g) = g^2$
Is there physics beyond BCS?

fixed point beyond FL?
NiO insulates $d^8$? perhaps this costs energy

$$U \gg t$$

**local real-space physics**

$$\mu = 0$$

no change in size of Brillouin zone

$Y_{Ba_{2}Cu_{3}O_{7}}$

Cuprate Superconductors
solve the Hubbard Model!!

Cooper instability??

Progress thus far?

DMFT  QMC  disputes
idea is either missing or improperly understood. Another indicator that something is deeply wrong is the inability of anyone to describe the elementary excitation spectrum of the Mott insulator precisely even as pure phenomenology. Nowhere can one find a quantitative band structure of the elementary particle whose spectrum becomes gapped. Nowhere can one find precise information about the particle whose gapless spectrum causes the paramagnetism. Nowhere can one find information about the interactions among these particles or of their potential bound state spectroscopies. Nowhere can one find precise definitions of Mott insulator terminology. The upper and lower Hubbard bands, for example, are vague analogues of the valence and conduction bands of a semiconductor, except that they coexist and mix with soft magnetic excitations no one knows how to describe very well.

The fundamental argument is presented in the second paragraph: “Ten years of work by some of the best minds in theoretical physics have failed to produce any formal demonstration”…of the Mott insulating state. The statement would be ludicrous if it were not so influential. The proviso “at zero temperature” is added, because of course most Mott concern. It is the tragedy of Mott that although he almost certainly won his Nobel prize for the Mott insulator, Slater, who couldn’t think clearly about finite temperature, won the publicity battle.
Laughlin’s objection:

gap with no symmetry breaking not demonstrated!!

charge gap

1 \rightarrow 2
Mottness zeros

\[ \text{DetReG}(\omega = 0, p) = 0 \]

\[ \neq 0 \]
counting particles

is there a more efficient way?
Luttinger counting theorem

\[ G(E) = \frac{1}{E - \varepsilon_p} \]

\[ n = 2 \sum_k \Theta(\Re G(k, \omega = 0)) \]

zero-crossing

\[ \det G(\omega = 0, \vec{p}) = 0 \]

counting poles (qp)
How do zeros obtain?

\[ \text{Re} G(0, p) = \int_{-\infty}^{\infty} \left( \mu = 0 \right) \text{d}\omega \]

= below gap + above gap

\[ \text{Im} G = 0 \]

\[ \text{Det} G(k, \omega = 0) = 0 \] (single band)

strongly correlated gapped systems
Symmetry Breaking

\[ G_k(\omega) = \begin{pmatrix} \frac{1}{\omega - E_k^+} & 0 \\ 0 & \frac{1}{\omega - E_k^-} \end{pmatrix} \]

\[ \text{Det} G \neq 0 \]

no Mottness

different band indices
are zeros important?
in a metal?
Fermi arcs necessarily imply zeros exist.

Must cross a zero line (\(\text{Det}G=0\))!!!
Where's Mottness??

Fermi Surface

irrelevant

What went wrong?
simplest model with zeros

atomic limit of Hubbard model

\[ H_{t=0} = -\mu \sum_{i\sigma} n_{i\sigma} + \sum_i U n_{i\uparrow} n_{i\downarrow} \]
Exact Green Function

\[ G_i^R(\omega) = \frac{1 + x}{\omega - \mu + \frac{U}{2}} + \frac{1 - x}{\omega - \mu - \frac{U}{2}}, \]

\[ x = 0 \]

\[ G_i^R(\omega = \mu = 0) = 0 \]

\[ n = 2\Theta \left( \frac{-\mu}{\mu^2 - U^2/4} \right) \]

LSR violation
Can the same physics be retained in the presence of a kinetic energy?
Mott insulator in momentum space???
Hatsugai-Khomoto Model (1992)

\[ H_{HK} = -t \sum_{\langle j,l \rangle, \sigma} \left( c_{j\sigma}^\dagger c_{l\sigma} + h.c. \right) - \mu \sum_{j\sigma} c_{j\sigma}^\dagger c_{j\sigma} \]

\[ c_{k\sigma} = \sum_{j} e^{ikj} c_{j\sigma} \]

\[ H_{HK} = \sum_{k} H_{k} = \sum_{k} \left( \xi_{k} (n_{k\uparrow} + n_{k\downarrow}) + U n_{k\uparrow} n_{k\downarrow} \right) \]

\[ \xi_{k} = \epsilon_{k} - \mu \]

relevant interaction
General HK Model

\[ \sum_k (\xi_k (n_{k\uparrow} + n_{k\downarrow}) + U n_{k\uparrow} n_{k'\downarrow}) \].

\[ [H_t, H_U] = 0 \]

Solvable Mott transition

\[ G_{k\sigma}(i\omega_n \to z) = \frac{1 - \langle n_{k\bar{\sigma}} \rangle}{z - \xi_k} + \frac{\langle n_{k\bar{\sigma}} \rangle}{z - (\xi_k + U)} \]

lower Hubbard band

upper Hubbard band
Hubbard band operators

\[ \zeta_{k\sigma} = c_{k\sigma}^\dagger \left(1 - n_{k\bar{\sigma}}\right) \]

\[ \eta_{k\sigma} = c_{k\sigma}^\dagger n_{k\bar{\sigma}} \]

\[ \langle n_{k\sigma} \rangle = \frac{1}{2} \]

\[ G^R_{\sigma}(k, \omega) = \frac{1}{\omega + i0^+ - (\xi_k + U/2) - \frac{(U/2)^2}{\omega + i0^+ - (\xi_k + U/2)}} \]

\[ \Sigma(k, \omega) \]

\[ \omega = \xi_k + U/2 \]

\[ \Re \Sigma = \Im \Sigma = \infty \]

Fermi liquid
QP
Mott transition: composite excitations

\[ \Delta E = U - 4dt = U - W \]

insulator

Luttinger Surface

metal
\[ \text{FL}^+ U n_k \uparrow n_{k'} \downarrow = M.I. \]
metal is degenerate: SU(2) invariance

\[ |\Psi_G; \{\sigma_k\} \rangle = \prod_{k \in \Omega_1} c_k^\dagger \sigma_k \prod_{k \in \Omega_2} c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger |0\rangle, \]
what does the HK model leave out??

\[ [H_t, H_U] \neq 0 \]

dynamical spectral weight transfer
Superconductivity?

C. Setty
2019

ILS+ Pair fluctuations = SYK

Is this true in general?
Cooper Instability

\[ H = H_{HK} - g H_p \]

\[ |\psi\rangle = \sum_{k \in \Omega_0} \alpha_k b_k^\dagger |\text{GS}\rangle \]

\[ \langle n_{k\sigma} \rangle = 0 \]

\[ E_b = \langle \text{GS}|H|\text{GS} \rangle - \langle \psi|H|\psi \rangle \leq 0 \]

\[ 1 = -\frac{g}{L^d} \sum_{k \in \Omega_0} \frac{\langle 1 - n_{k\uparrow} + n_{-k\downarrow} \rangle}{E - 2\xi_k - U \langle n_{k\downarrow} + n_{-k\uparrow} \rangle} \]

\[ 1 = -g \int_{\mu}^{W/2} d\epsilon \frac{\rho(\epsilon)}{E - 2\epsilon + 2\mu}, \]
$E_b = -E \sim W(1 - (U/W)^2)e^{-\pi W \sqrt{1-(U/W)^2/g}}$
Pair Susceptibility

\[ \chi(i\nu_n) = \frac{1}{L^d} \int_0^\beta d\tau e^{i\nu_n\tau} \langle T\Delta(\tau)\Delta^\dagger \rangle_g \]

\[ = \frac{\chi_0}{1 - g\chi_0} \]

\[ \sum_k G_{-k\downarrow}(\tau)G_{k\uparrow}(\tau) \]

solve for \( T_c \)
\[ T_c = (W - U)^{4/5} U^{1/5} \frac{e^\gamma}{\pi} e^{-\frac{4}{5} \frac{W}{g}}. \]
$$|\psi_{\text{BCS}}\rangle = \prod_k (u_k + v_k b_k^\dagger |0\rangle$$

$$|\psi_{\text{BCS}}\rangle = \prod_{k>0} (u_k^2 + v_k^2 b_k^\dagger b_{-k}^\dagger + u_k v_k (b_k^\dagger + b_{-k}^\dagger)) |0\rangle$$
\[ |\psi\rangle = \prod_{k>0} \left( x_k + y_k b_k^\dagger b_{-k}^\dagger + \frac{z_k}{\sqrt{2}} (b_k^\dagger + b_{-k}^\dagger) \right) |0\rangle \]
three variational parameters

\[ |x_k|^2 + |y_k|^2 + |z_k|^2 = 1 \]

gap equation

\[ 1 = \frac{g}{W} \sinh^{-1}\left( \frac{W - U}{2\Delta} \right) + \frac{g}{W} \sinh^{-1}\left( \frac{U}{2\Delta} \right) \]

\[ \Delta \ll U, W \]

\[ \Delta = (W - U)^{1/2} U^{1/2} e^{-\frac{W}{2g}} \]
\[
\Delta = (W - U)^{1/2} U^{1/2} e^{-\frac{W}{2g}}
\]

\[
T_c = (W - U)^{4/5} U^{1/5} \frac{e^\gamma}{\pi} e^{-\frac{4}{5} \frac{W}{g}}.
\]

\[
\lim_{g \to 0} \frac{\Delta}{T_c} \to \infty
\]

non-BCS superconductivity
Bogoliubov excitations

\[ \gamma_{k\sigma} |\psi_{BCS}\rangle = 0 \]

\[ \gamma_{k\sigma} = u_k \zeta_{k\sigma}^\dagger - \sigma v_k \zeta_{-k\bar{\sigma}} \]

PYHons excitations

\[ \gamma^l_{k\sigma} \propto \sqrt{2} x_k \zeta_{k\sigma}^\dagger - \sigma z_k \zeta_{-k\bar{\sigma}} \]

\[ \gamma^u_{k\sigma} \propto z_k \eta_{k\sigma}^\dagger - \sigma \sqrt{2} y_k \eta_{-k\bar{\sigma}} \]
Excitation spectrum

$$\gamma^{u/l}_{k\sigma} |\psi\rangle = 0$$

$$\langle \psi | \gamma^{u/l}_{k\sigma} H \gamma^{u/l}_{k\sigma} \rangle^{\dagger} |\psi\rangle = \langle \psi | H |\psi\rangle + E^{u/l}_k$$

$$E^{u/l}_k = \sqrt{\xi^{u/l}_k^2 + \Delta^2}$$

superconductivity affects both bands!
spectral function: superconducting state

Doped MI

half-filled metal
can we explain the color change?

REPORT

Superconductivity-Induced Transfer of In-Plane Spectral Weight in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$

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\[ A_l = \int_0^\Omega \sigma(\omega) d\omega \quad \Omega/2\pi c = 10000 \text{cm}^{-1} \]

\[ A_h = \int_{\Omega}^{2\Omega} \sigma(\omega) d\omega \quad \Omega/2\pi c = 10000 \text{cm}^{-1} \]

\[ \frac{\Delta A_l}{A_l} \propto 3\% \]
Optical data are reported on a spectral weight transfer over a broad frequency range of $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$, when this material became superconducting. Using spectroscopic ellipsometry, we observed the removal of a small amount of spectral weight in a broad frequency band from $10^4$ cm$^{-1}$ to at least $2 \times 10^4$ cm$^{-1}$, due to the onset of superconductivity. We observed a blue shift of the $ab$-plane plasma frequency when the material became superconducting, indicating that the spectral weight was transferred to the infrared range. Our observations are in agreement with models in which superconductivity is accompanied by an increased charge carrier spectral weight. The measured spectral weight transfer is large enough to account for the condensation energy in these compounds.
condensation energy: HK model

\[ \Delta W_L \propto O(2 - 3\%) \]

as in experiments
why?

\[ H = H_{HK} + H_p \]

\[ [H_{HK}, H_p] \neq 0 \]

dynamical spectral weight transfer
is this the general mechanism of the color change?
Superfluid Density

Mottness-induced suppression
Mott gap

\([H_t, H_V] = 0\)

Mottness in momentum space

\[ U n_{k \uparrow} n_{k' \downarrow} \]

PYHons

\[ \begin{align*}
\gamma^l_{k \sigma} & \propto \sqrt{2} x_k \zeta^\dagger_{k \sigma} - \sigma z_k \zeta_{-k \sigma} \\
\gamma^u_{k \sigma} & \propto z_k \eta^\dagger_{k \sigma} - \sigma \sqrt{2} y_k \eta_{-k \sigma}
\end{align*} \]

PYHons

non-BCS superconductivity

\[ \frac{\Delta}{T_c} \propto \infty \]

\[ \frac{W_L(g)}{W_L(g = 0)} > 1 \]