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Physics 485

Useful Mathematical Results

Algebra

1. (Euler identity) $\exp(i\phi) = \cos\phi + i\sin\phi$
2. $|A+B|^2 = |A|^2 + |B|^2 + 2\mathrm{Re}\{A^*B\} = |A|^2 + |B|^2 + 2\mathrm{Re}\{B^*A\}$
3. (Distribution) $\left(\sum_i a_i\right)\left(\sum_j b_j\right) = \sum_i\sum_j a_i b_j$

Commutators:

$\left[\breve{A}\breve{B},\breve{C}\right] = \breve{A}\left[\breve{B},\breve{C}\right] + \left[\breve{A},\breve{C}\right]\breve{B}$; $\left[\breve{A},\breve{B}\breve{C}\right] = \breve{B}\left[\breve{A},\breve{C}\right] + \left[\breve{A},\breve{B}\right]\breve{C}$; $\frac{d\langle\breve{Q}\rangle}{dt} = \frac{i}{\hbar}\left\langle\left[\breve{H},\breve{Q}\right]\right\rangle + \left\langle\frac{\partial\breve{Q}}{\partial t}\right\rangle$

$\left[\breve{p},x\right] = \frac{\hbar}{i}$; For $\breve{H} = \frac{\breve{p}^2}{2m} + V(x)$: $\left[\breve{H},x\right] = \frac{\hbar}{i}\frac{\breve{p}}{m}$; $\left[\breve{H},\breve{p}\right] = -\frac{\hbar}{i}\frac{\partial V}{\partial x}$; Virial Thm for S.S. : $2\langle T\rangle = \left\langle x\frac{\partial V}{\partial x}\right\rangle$

Inner Product Space

4. (Prototypes) $\langle\psi|\varphi\rangle = \int_{-\infty}^{+\infty} dx\,\psi^*(x)\ \varphi(x)$; $\langle\psi|\breve{G}|\varphi\rangle = \int_{-\infty}^{+\infty} dx\,\psi^*(x)\ \breve{G}\{\varphi(x)\}$
5. (Conjugation) $\langle\psi|\varphi\rangle^* = \langle\varphi|\psi\rangle$; $\langle\psi|\psi\rangle = \|\psi\|^2 > 0$; when normalized $\langle\psi|\psi\rangle = 1$
6. (Distribution) $\langle\psi|a\varphi_a + b\varphi_b\rangle = a\langle\psi|\varphi_a\rangle + b\langle\psi|\varphi_b\rangle$; $\langle a\varphi_a + b\varphi_b|\psi\rangle = a^*\langle\varphi_a|\psi\rangle + b^*\langle\varphi_b|\psi\rangle$

7. (Self-adjoint Operator Distribution)
$\langle a\varphi_a + b\varphi_b|\breve{O}|a\varphi_a + b\varphi_b\rangle = |a|^2\langle\varphi_a|\breve{O}|\varphi_a\rangle + |b|^2\langle\varphi_b|\breve{O}|\varphi_b\rangle + 2\mathrm{Re}\{a^*b\langle\varphi_a|\breve{O}|\varphi_b\rangle\}$
8. (Completeness) $\{\varphi_i\}$ of all φ_i : $\hat{G}\varphi_i = g_i\varphi_i$; $\sum_i|\varphi_i\rangle\langle\varphi_i| = 1 \rightarrow |\psi\rangle = \sum_i|\varphi_i\rangle\langle\varphi_i|\psi\rangle$; $\mathcal{P}(g_i) = |\langle\varphi_i|\psi\rangle|^2$
9. (Adjoint) $\langle\psi|\breve{G}|\varphi\rangle = \langle\psi|\breve{G}\{\varphi\}\rangle = \langle\breve{G}^\dagger\{\psi\}|\varphi\rangle$ Observables are self-adjoint: $\breve{G} = \breve{G}^\dagger \rightarrow \langle\psi|\breve{G}|\psi\rangle \in \Re$

10. (Statistics) $\langle G\rangle = \langle\psi|\breve{G}|\psi\rangle$; $\sigma^2(G) = \langle G^2\rangle - \langle G\rangle^2 = \langle\psi|\breve{G}^2|\psi\rangle - \langle\psi|\breve{G}|\psi\rangle^2 \ge 0$

Direct Product Space

11. (Prototype direct prod and operators) $|\psi\rangle = \varphi(x)\times\eta(y) \equiv |\varphi\,\eta\rangle$; $\breve{\mathrm{p}}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$; $\breve{\mathrm{p}}_y = \frac{\hbar}{i}\frac{\partial}{\partial y}$
12. (Prototype example)

$$\langle\psi'|\breve{p}_x\breve{p}_y|\psi\rangle = \langle\varphi'\eta'|p_x p_y|\varphi\,\eta\rangle$$
$$= \left[\int_{-\infty}^{+\infty} dx\ [\varphi'(x)]^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\{\varphi(x)\}\right)\right]\times\left[\int_{-\infty}^{+\infty} dy\ [\eta'(y)]^*\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\{\eta(y)\}\right)\right]$$
$$= \langle\varphi'|\breve{p}_x|\varphi\rangle\langle\eta'|\breve{p}_y|\eta\rangle = \langle\varphi'|\breve{p}|\varphi\rangle\langle\eta'|\breve{p}|\eta\rangle$$

13. (Direct product factorization)
Let $|\psi\rangle = |\varphi\,\eta\rangle$ be $\varphi\otimes\eta$ direct prod, and $\breve{G} = \breve{A}_\varphi\breve{B}_\eta$ where $\breve{A}_\varphi$ only operates on φ and $\breve{B}_\eta$ only operates on η, then:
$\langle\psi'|\breve{G}|\psi\rangle = \langle\psi'|\breve{A}_\varphi\breve{B}_\eta|\psi\rangle = \langle\varphi'\eta'|\breve{A}_\varphi\breve{B}_\eta|\varphi\,\eta\rangle = \langle\varphi'|\breve{A}|\varphi\rangle\langle\eta'|\breve{B}|\ \eta\rangle$

Identical particle notation and identity

One particle is in state m and the other in state n and $m \neq n$ and the spatial wave functions are either symmetric or anti-symmetric depending on spin

$$\psi(x_1,x_2) = \frac{|mn\rangle \pm |nm\rangle}{\sqrt{2}} \equiv \frac{\varphi_m(x_1)\varphi_n(x_2) \pm \varphi_n(x_1)\varphi_m(x_2)}{\sqrt{2}}$$

We can distribute an operator $\breve{O}$ using this identity

$$\langle\psi|\breve{O}|\psi\rangle = \frac{\langle mn \pm nm|\breve{O}|mn \pm nm\rangle}{2} = \frac{\langle mn|\breve{O}|mn\rangle + \langle nm|\breve{O}|nm\rangle}{2} \pm \mathrm{Re}\left\{\langle mn|\breve{O}|nm\rangle\right\}$$

Useful Integrals

1. (Exponential integral)

$$\int_0^{+\infty} dx\ \mathrm{x}^{\mathrm{n}} \exp(-x) = n!$$

2. (Gaussian full integrals)

$$I_n = \int_{-\infty}^{+\infty} dx\ \mathrm{x}^{\mathrm{n}} \exp(-\beta x^2)\ ;\ I_{n+2} = -\frac{\partial I_n}{\partial \beta}\ \ ;\ \ I_0 = \sqrt{\pi}\beta^{-1/2}$$

$$I_2 = \sqrt{\pi}\left(\frac{1}{2}\right)\beta^{-3/2}\ ;\ \ I_4 = \sqrt{\pi}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\beta^{-5/2}\ ;\ I_6 = \sqrt{\pi}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\beta^{-7/2}$$

3. (Gaussian half integrals)

$$J_n = \int_0^{+\infty} dx\ \mathrm{x}^{\mathrm{n}} \exp(-\beta x^2)\ ;\ \mathrm{J}_{n+2} = -\frac{\partial J_n}{\partial \beta}\ \ ;\ \ J_0 = \sqrt{\pi}\left(\frac{1}{2}\right)\beta^{-1/2}\ ;\ J_1 = \left(\frac{1}{2\beta}\right)$$

$$J_2 = \sqrt{\pi}\left(\frac{1}{4}\right)\beta^{-3/2}\ ;\ \ J_3 = \left(\frac{1}{2}\right)\beta^{-2}$$