Topological Quantum Computing in (p + ip) Fermi Superfluids:

Some Unorthodox Thoughts

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General principle of topological quantum computing (TQC): within relevant Hilbert space,

1. relevant quantum states indistinguishable by any "natural" operation $\left(\hat{H}_{nat} \propto \hat{1}\right)$

2. Can perform necessary unitary transformations by "unnatural" operations (which should be robust)

Proposal: in 2D (p + ip) Fermi superfluids, perform TQC by braiding vortices containing Majorana fermions

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2D p + ip Fermi superfluids (thin slabs of ^{3}HeA, Sr_2RuO_4, UC Fermi gases…): order parameter in uniform case has form (**p** near Fermi surface)

$$\Delta(\boldsymbol{p}) = \Delta_O\left(p_x + ip_y\right)$$

Note: (a) in 2D, $\left|\Delta(p)\right| = \text{const.}$

(b) breaks time-reversal symmetry

(c) Pauli principle $\Rightarrow$ parallel spins, e.g. $|\uparrow\uparrow\rangle$

Textbook approach to fermionic excitations in superconductors (Fermi superfluid):

invoke spontaneous braking of U(1) symmetry (SBU(1)S):

$$\Psi_{even} = \sum_N C_{2N} \Psi_{2N} \quad \leftarrow \text{Not eigenstate of particle no.}$$

then simplest fermionic excitation is quantum superposition of extra particle and extra hole (Bogoliubov quasiparticle)

e.g. for uniform case,

$$\alpha_k^+ = u_k a_k^+ + \upsilon_k a_{-k}$$

In general case (nonuniform gap) creation operator of Bogoliubov quasiparticle is

$$\alpha_i^+ = \int dr \left\{ u(r) \underset{\substack{\uparrow \\ \text{Extra particle}}}{\hat{\psi}^\dagger(r)} + \upsilon(r) \underset{\substack{\uparrow \\ \text{Extra hole}}}{\hat{\psi}(r)} \right\}$$

with $\left(u(r), \upsilon(r)\right)$ solution of Bogoliubov-de Gennes equations

$\left[H_{BdG}, \alpha_i^+ \right] = E_i \alpha_i^+$ i.e.

$$\hat{H}_{BdG} \begin{pmatrix} u(r) \\ \upsilon(r) \end{pmatrix} \equiv \begin{pmatrix} \hat{H}_O & \overset{\text{gap}}{\Delta(r)} \\ \Delta^*(r) & -\hat{H}_O^* \end{pmatrix} \begin{pmatrix} u(r) \\ \upsilon(r) \end{pmatrix} = E \begin{pmatrix} u(r) \\ \upsilon(r) \end{pmatrix}$$

Single-particle energy

Majorana fermion:

Solution to BdG with properties

(1) $u(r), \upsilon(r)$ localized in space
(2) $E = 0$
(3) $u(r) = \upsilon^*(r) \Rightarrow \gamma_i^\dagger \equiv \gamma_i$ (Majorana, 1937)

Because of (3), Majoranas undetectable by any local probe (condition (1)). Moreover, under braiding (robust procedure) form representation of braid group (condition (2)).

In (p + ip) superfluids, a (half-quantum) vortex/antivortex admits exactly one Majorana solution $\Rightarrow$ MF solutions always come in pairs.

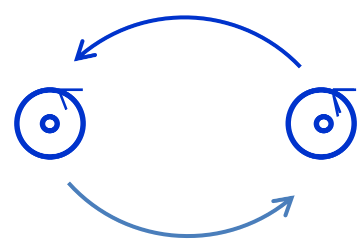
Proposal*:

(1) create vortex-antivortex pair { without Bog. qp/ No MF's
with Bog. qp MF on each

(2) braid (permute) vortices and antivortices

(3) recombine, read off presence/absence of Bog. qp's.

should realize (Ising) TQC.
Simplest case: exchange of 2 vortices
Prediction for Berry phase φ_B (Ivanov):
for no MF's, $\varphi_B = 0$
for 2 MF"s, $\varphi_B = \pi/2$



(note not usual fermionic π!)

i.e.

$$\hat{U}_{\text{exch}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$

*Ivanov, PRL **86**, 268 (2001): Stone & Chung, PRB **73**, 14505 (2006)

Digression: what exactly is a Majorana fermion?

By definition, it is a solution of the BdG equations

$$\left[H,\gamma_i^{\dagger}\right]=E\gamma_i^{\dagger} \qquad \text{with } E = 0$$

But this has two possible interpretations when acting on the even-parity groundstate:

(a) $\gamma_i^{\dagger}$ creates an extra Bogoliubov quasiparticle with zero energy

(b) $\gamma_i^{\dagger}$ simply annihilates the groundstate ("pure annihilator")

But in neither case can $\gamma_i^{\dagger}$ be equal to γ_i! To ensure this we must superpose (a) and (b) with equal weight, i.e.

A Majorana fermion is a quantum superposition of a zero-energy fermion and a pure annihilator

The physically real object is the E = 0 fermion, which is a quantum superposition of two Majoranas:

$$\alpha^{+}=\gamma_i^{\dagger}+i\gamma_2^{\dagger} \qquad \text{(always possible since MF's come in pairs)}$$

If e.g. $\gamma_1^{\dagger},\gamma_2^{\dagger}$ refer to 2 vortices in a (p + ip) superfluid, the zero-energy fermion is strongly delocalized ("split")

($\Rightarrow$ teleportation? Controversial!)

(So far, in some sense well-known…)

A slight problem with the "standard" approach:

SBU(1)S IS A MYTH!

(The states of real systems are always either eigenstates of particle number $\hat{N}$, or incoherent mixtures thereof)

Suppose we start (say) in the even-no.-parity groundstate $|2N, 0\rangle$. Then "textbook" prescription for Bogoliubov quasiparticle operator α_k^+ gives

$$\alpha_k^+ |\Psi_{2N}\rangle \equiv \left(u_k a_k^+ + \upsilon_k a_{-k}\right) = u_k \underbrace{|2N+1, k\rangle + \upsilon_k |2N-1, k\rangle}_{\substack{\uparrow \\ \text{not number-conserving}}}$$

So, we need to write instead

$$\alpha_k^+ = u_k a_k^+ + \upsilon_k a_{-k} \hat{C}^\dagger \qquad \leftarrow \text{Cooper-pair creation operator}$$

Q: How come we (mostly*) got away with ignoring this for 50 years?

A: As long as Cooper pairs carry no interesting quantum numbers, doesn't matter! However, once they have nonzero COM, spin..., this becomes crucial and standard "mean-field" ideas may fail.

Examples:

(a) NMR in ^{3}He-B (C. pairs have nonzero spin)

(b) Galilean invariance (C. pairs have nonzero COM momentum)

Now: in a *p* + *ip* superfluid, C. pairs have "internal" angular momentum!

So: are standard mean-field ideas adequate for quantum-information purposes (in particular, TQC)?

*But cf. e.g. Blonder at al., PRB **25**, 4515 (1982)

Some Simple Consequences of Cooper Pair Angular Momentum

1. Why does a vortex in a *p* + *ip* superfluid, but not in an s-wave one, carry Majoranas?

$$\gamma_0^\dagger = u(r)\hat{\psi}^\dagger(r) + \upsilon(r)\hat{\psi}(r)\hat{C}$$

$u(r), \upsilon(r)$ single-valued $\Rightarrow$ characterized by angular momentum QNS $\ell_u, \ell_\upsilon = 0, \pm1, \pm2 \ldots$

$u(r) = \upsilon^*(r) \Rightarrow \ell_u = -\ell_\upsilon$

Suppose "local" angular momentum of Cooper pair is $\ell_c^{(loc)}$, then conservation of total angular momentum

$$\Rightarrow \ell_u = \ell_\upsilon + \ell_c^{(loc)} \Rightarrow \ell_c^{(loc)} = 2\ell_u = \text{even.}$$

ℓ_c has contribution from COM (vortex) and possibly intrinsic angular momentum.

In s-wave, $\ell_{\text{vort}} = (\pm)1, \ell_{\text{int}} = 0 \Rightarrow \ell_c^{loc} = \text{odd} \Rightarrow$ MF's cannot exist.

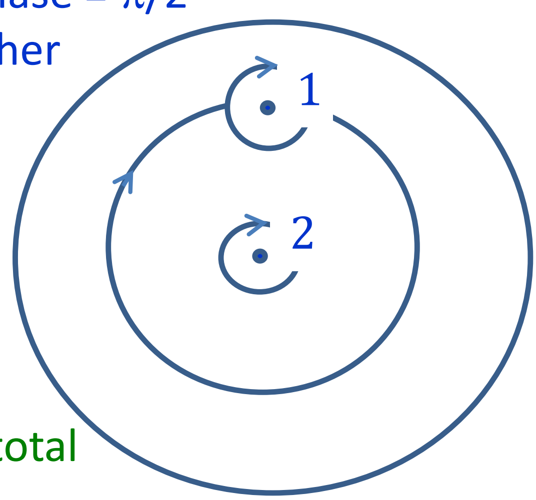
In *p* + *ip*, $\ell_{\text{vort}} = (\pm)1, \ell_{\text{int}} = 0 \Rightarrow \ell_c^{loc} = \text{odd} \Rightarrow$ MF's may exist.

(need further argument to show that they do).

2. Exchange of two vortices with/without MF's:

recall: acc. Ivanov, relative Berry phase = π/2

Consider encirclement of one by another



Theorem (YRL): In this situation,
encirclement Berry phase $= 2\pi \bullet \langle L \rangle$

↑

expectation value of total angular momentum

Recap: $\varphi_B^{(ene)} = 2\pi \langle L \rangle$.

1. No MF's:

$\langle L \rangle = N_c \cdot \ell_c = \text{integral}$

$\Rightarrow$ encirclement phase $= 2n\pi = 0 \bmod 2\pi$

$\Rightarrow$ exchange phase $= 0$ (or π , but exclude on physical grounds)

2. MF's on vortices 1 and 2 (i.e. $E = 0$ fermion "split" between 1 and 2).
What is extra Berry phase? i.e. what is $\Delta\langle L \rangle$?

(a) "standard" approach:

$\left\{\begin{array}{l}\text{Angular momentum of M.F.'s themselves} \equiv 0 \\ \text{no change in Cooper pair state}\end{array}\right.$ (otherwise locally detectable!)

$\Rightarrow \Delta \langle L \rangle = 0 \Rightarrow$ encirclement phase $= 0 \Rightarrow$ exchange phase $= 0$ or π
(not adequate for TQC)

(b) Number–conserving approach:

One extra Cooper pair is added in conjunction with $\upsilon(r)$, i.e. exactly half the time. Hence

$\Delta\langle L \rangle = \frac{1}{2}\ell_c$ where ℓ_c is global C. pair angular momentum

But for 2 vortices (or an even number)

$\ell_{\text{c, vort}} = \text{even}, \ell_{\text{int}} = \pm 1$

$\Rightarrow \ell_c$ odd $\Rightarrow$ encirclement phase $= (2n + 1)\pi$

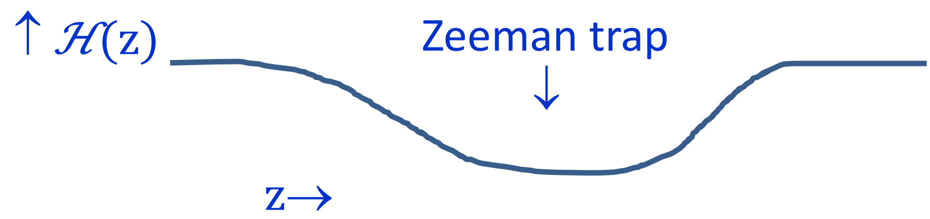
$\Rightarrow$ Exchange phase is $\pi/2$ (mod. π)

– the Ivanov result!

Yet… Ivanov's argument on exchange never invokes p-wave nature of OP!

Two \$64 questions beyond the mean-field scenarios

(1) Is the “extra” Cooper pair exactly the same as the pre-existing ones? e.g.



add one up-spin Bogoliubov or quasiparticle:
everyone agrees $\Delta S_{\mathrm{loc}} \cong 1.$
What about ΔN_{loc} ?
Mean-field answer: $\Delta N_{\mathrm{loc}} = 0$
but is “0” $1/N_{\mathrm{tot}}$ or $1/N_{\mathrm{trap}}$? (In latter case, may be inadequate for TQC)

(2) The BdG equations relate (the simplest) even– and odd–number parity many-body states. i.e. if we have (e.g.) the even-parity and odd-parity groundstates, then

$$\Psi_{\mathrm{odd}} = \alpha_0^+ \Psi_{\mathrm{even}}$$

$$\alpha_0^+ \equiv \int \left\{ u_0(r)\psi^\dagger(r) + \upsilon(r)\psi(r) \right\} dr$$

$(u(r), \upsilon(r))$ solution of BdG equations.

Question: IS THE CONVERSE TRUE?

i.e. does the existence of a solution (u_0, υ_0) to the BdG equations imply that there exist even– and odd–parity states connected by it?