The Spin Dynamics of an Anisotropic Fermi Superfluid ($^3$He?)

A. J. LEGGETT*

Laboratory of Atomic and Solid State Physics,
Cornell University, Ithaca, New York 14850 and
School of Mathematical and Physical Sciences,
University of Sussex, Falmer, Brighton BN1 9QH
Sussex, UK † and Department of Physics, Faculty of Science,
University of Tokyo, Bunkyo-ku, Tokyo 113, Japan

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In this paper we develop a general theory of the spin dynamics of anisotropic Fermi superfluids of the generalized BCS type, under conditions which should be realistic for any such phase of liquid $^3$He occurring below 3 mK. No restrictions are placed on the nature of the pairing configuration. The system is described in terms of the total spin vector $S$, and a vector $T(n)$ which describes the amplitude and spin quantization axes of the pairs forming at a given point $n$ on the Fermi surface; the kinematic relations between these quantities are emphasized. An approximation of the Born-Oppenheimer type is used to derive the general equations of motion of $S$ and $T$; it is pointed out that relaxation of $T$ due to collisions is inhibited by the coherent nature of the superfluid state. The equations of motion are solved for the particular case of unsaturated c.w. resonance, and it is shown that the nature of the transverse (usual) resonance spectrum is a strong function of the kind of configuration occurring; in particular, either one or two finite-frequency resonances may occur, depending on the configuration. A resonance is also predicted to occur when the r.f. field is polarized along the static external field. Specific predictions of the form of the transverse and "longitudinal" spectra are made for all the unitary $l = 1$ states, and it is shown that these predictions are unaffected by renormalization effects. The "Balian-Werthamer" state is predicted to show a longitudinal resonance but no transverse shift. The theory is compared with other approaches to the problem and its relevance to the anomalous low-temperature phases of liquid $^3$He is discussed.

1. INTRODUCTION

In the last year or so it has become clear that liquid $^3$He possesses, apart from the familiar "normal" phase which seems to be relatively well understood in terms of Landau's theory of a Fermi liquid [1], at least two anomalous phases which occur only below 3 mK. Of the many interesting experimental results already known about these phases, one of the most intriguing remains the

† Permanent address.
remarkable shift of the nuclear magnetic resonance frequency observed [2] in the higher-temperature A phase. It seems that as the liquid is cooled through the A transition, the spin susceptibility is little, if at all, affected [2, 3] but the nuclear resonance signal shifts away from the “free” Larmor frequency \( \omega_L = \gamma H_0 \) (\( H_0 \) = external field) to higher values; the magnitude of the shift is not proportional to \( \omega_L \) but seems to be well described by the formula [2]

\[
\omega^2 = \omega_L^2 + \omega_0^2(T),
\]

where \( \omega \) is the observed signal frequency and \( \omega_0 \) is not a function of \( H_0 \). However, it should be noted that the values of the field used in experiments to date were such that \( \omega_0(T) < \omega_L \) even at the lowest temperatures; thus, we cannot necessarily infer that formula (1.1) is valid for all \( \omega_L \).

It was shown in a previous paper by the author [4] that the shift could be qualitatively understood if liquid \(^3\)He-A possesses the property of “spontaneously broken spin–orbit symmetry” (SBSOS); this concept was explored more fully in [5], to which we refer for details. However, without a detailed microscopic model of the A phase it is impossible to say, for example, whether Eq. (1.1) will extrapolate to zero \( \omega_L \), what information may be obtained from possible NMR experiments of less conventional types, or whether one can draw any positive conclusions about the B phase from the fact that it does not show a shift.

At the time of writing, at least, the most plausible microscopic model for the A phase seems to be an anisotropic BCS-like phase; that is, a generalization of the BCS state proposed [6] for the electrons in a superconductor to the case when the Cooper pairs form with nonzero orbital angular momentum. That liquid \(^3\)He should indeed make a transition to such a condensed phase was predicted by a number of people many years ago [7–9]; it is generally supposed that it would then show the property of superfluidity, and without wishing to beg this question we shall therefore call such a state, for brevity, an anisotropic superfluid. It is clear that an anisotropic superfluid does in general possess the SBSOS property required to account qualitatively for the NMR shift, provided that the Cooper pairs form with total spin one and hence with odd orbital angular momentum \( I \); moreover, the unchanged susceptibility of the A phase is in accordance with this model, if we assume that the Cooper pairs form only in states with \( S_z = \pm 1 \) and not also with \( S_z = 0 \) (i.e., in a so-called “ESP” state—see below). Consequently, it has been widely assumed that the A phase is indeed an odd-\( I \), ESP-type anisotropic superfluid [4], and this hypothesis seems to be in at least qualitative agreement [10, 11] with most other experimental results known to date [12–16]. Some theoretical difficulties associated with the stability of an ESP phase [17] have received a possible resolution in the work of Anderson and Brinkman [18], who have also proposed that the B phase may also be a triplet superfluid of a different type.
In this paper we shall develop a general theory of NMR phenomena in an anisotropic superfluid, which we believe will be adequate to describe the behaviour of either or both new phases of liquid $^3$He if they are indeed of this type. Since in an even-$l$ state the Cooper pairs have spin zero, such a state does not have the SBSOS property and the results then are just the normal ones [4] (apart from questions connected with the detailed temperature-dependence of linewidths, etc.); we shall therefore confine attention to the case of odd-$l$, $S = 1$ pairing. However, in the main body of this paper we shall make no further assumptions about the nature of the pairing configuration; our results will therefore apply equally to ESP and non-ESP states, to pairs forming with any $l$-value (or indeed a mixture of different $l$ values) and irrespective of whether effects of the type emphasized by Anderson and Brinkman [18] are important. While the theory is in principle applicable to very general types of NMR experiment, we work it out in detail for the important case of unsaturated c.w. resonance; the results are then expressed in terms of certain characteristic constants which have very different values depending on the particular configuration involved. We shall show in fact that by NMR experiments alone it is possible to place very severe constraints on the nature of the configuration occurring in $^3$He-A, and also to test whether $^3$He-B is indeed a spin-1 anisotropic superfluid.

The plan of the paper is as follows. In Section 2 we discuss briefly the general problem of $S = 1$ pairing, and introduce a convenient set of macroscopic variables to describe the system; we also derive the commutation relations of these variables, which play an essential role in the subsequent argument. In Section 3 we discuss the nature of the thermodynamic equilibrium state, taking careful account of the dipole forces as is essential [5]. Section 4 introduces the physical approximation which is the basis of the theory developed here, discusses its region of validity and some possible objections to it, including the question of relaxation, and uses it to derive the general equations of motion which describe the spin dynamics of the system. In Section 5 we specialize to the case of unsaturated c.w. resonance in the simplest case (where "susceptibility anisotropy" can be neglected) and solve the equations of motion to get the full dynamic spin susceptibility of the system. In Section 6 we take up the question of "susceptibility anisotropy" and derive the corrections to the formulae of Section 5 which it induces; we find that these are finite only for a limited class of states. The results of Sections 5 and 6 are expressed in terms of certain characteristic constants, and in Section 7 we evaluate these for the simplest case ($l = 1$ pairing with the "unitary" restriction—see below). We also comment on the absolute magnitude and temperature-dependence of the NMR shift and tabulate the characteristic behaviour of the various states discussed. In Section 8 we compare our results with those of other authors who have dealt with the problem. In Section 9 we discuss the physical significance of our theory and its implications for the new phases of liquid $^3$He. An appendix deals with some
mathematical details relevant to the evaluation of the characteristic constants; in particular, it is shown that for $P$-states at least their ratios are not affected by renormalization effects.

Most of the main results of Sections 2–5 and 7 have been already given in condensed form [19], but without detailed discussion or justification.

2. THE KINEMATICS OF ANISOTROPIC SUPERFLUIDS

A convenient language for the description of anisotropic superfluids in which the Cooper pairs form with total spin one may be based on the work of Balian and Werthamer [17] (hereafter referred to as BW). Although these authors were specifically interested in pairing with $I = 1$, much of their discussion may be carried over virtually unchanged to the case of pairing with other odd values of $I$.

BW noted that the earlier treatments of $S = 1$ superfluids (e.g. [7–9] and [20]) had considered only pairing of “up” spins among themselves and “down” spins among themselves; this is equivalent to considering only the $S_z = \pm 1$ components of the spin triplet state $S = 1$, $S_z = -1, 0, +1$. A state in which only such pairs were formed was called by them an “equal-spin-pairing” (ESP) state. They then pointed out that it is essential also to take into account the possibility of symmetric pairing of up and down spins, i.e., the $S_z = 0$ component of the triplet. In fact they showed that for $I = 1$ pairing (in the weak-coupling approximation and in the absence of a magnetic field) the lowest free energy was always obtained by forming all three components of the triplet (in equal measure), and they gave a suitable formalism for doing this which has been the basis of most subsequent work on the equilibrium properties. Actually, in the case of liquid $^3$He-A it seems probable that the equilibrium state is of the ESP type, possibly being stabilized by strong-coupling effects [18]. Nevertheless, if we want to describe the NMR behavior it is essential to take into account the $S_z = 0$ component into account, since in NMR the spins are tipped away from the $z$-axis, which is equivalent to mixing into their wavefunction some of this component. Consequently we use from the outset a formalism which is closely related to BW’s, and like theirs, does not distinguish the $z$-axis in any special way; it then becomes irrelevant whether our equilibrium state is of ESP type or not.

We consider the pair of plane-wave states associated with momentum $p$ and $(-p)$ and define the following operators:

$$S_i(p) \equiv \text{Tr} a_p^+ \sigma_i a_p = \sum_{a\bar{a}} a_{p a} (\sigma_i)_{a\bar{a}} a_{p\bar{a}}, \quad (2.1)$$

$$T_i(p) \equiv \text{Tr} a_{-p}^+ \sigma_i a_p = \sum_{a\bar{a}} a_{-p a} (\sigma_i)_{a\bar{a}} a_{p\bar{a}}, \quad (2.2)$$
where the $\sigma_i$ are Pauli matrices ($i = 0, 1, 2, 3$) with $\sigma_0$ the unit matrix, and the creation and annihilation operators $a_{p\alpha}^+$, $a_{p\beta}$ are to be taken for the moment as referring to real particles (but see Section 7 below). From the definitions (2.1) and (2.2) follow the properties:

$$T_i(p) = \text{Tr} \, a_{p}^+ \sigma_i a_{-p}$$

(2.3)

$$T_i(p) = -\eta_i T_i(-p) \quad (\eta_i = 1 - 2\delta_{i,0}).$$

(2.4)

$$a_{-p\alpha}a_{p\beta} = \frac{1}{2} \sum_{i=0}^{3} (\sigma_i \sigma_2)_{\beta\alpha} T_i(p),$$

(2.5)

and the commutation relations

$$[S_i(p), S_j(p')] = \begin{cases} 2i\delta_{p\mu} \epsilon_{ijk} S_k(p) & i, j \neq 0 \\ 0 & i \text{ or } j = 0 \end{cases}$$

(2.6)

$$[S_i(p), T_j(p')] = \Delta_{p\mu}^j \zeta_{ijk} T_k(p),$$

(2.7)

$$[S_i(p), T_j(p')] = -\Delta_{p\mu}^i \zeta_{ijk} T_k(p'),$$

(2.8)

$$[T_i(p), T_j(p')] = \Delta_{p\mu}^i \zeta_{ijk}(S_k(p) + S_k(-p)) + 2\delta_{ij},$$

(2.9)

$$[T_i(p), T_j(p')] = [T_i^+(p), T_j^+(p')] = 0,$$

(2.10)

where for brevity we have introduced the notation

$$\Delta_{p\mu}^j = \delta_{p\mu} - \eta_j \delta_{p, -p'},$$

(2.11)

$$\zeta_{ijk} = \begin{cases} -\delta_{ik,0} & \text{if } i, j = 0 \\ -\delta_{jk} & \text{if } i = 0, \ j \neq 0 \\ -\delta_{ik} & \text{if } j = 0, \ i \neq 0 \\ i\epsilon_{ijk} - \delta_{ij}\delta_{k0} & \text{if } i, j \neq 0 \end{cases}$$

(2.12)

The commutation relations (2.6)–(2.10) play a fundamental role in what follows.

Evidently, the "zero" components $S_0$ and $T_0$ may be discussed separately from the 1, 2, 3, elements, which may be regarded as the components of three-vectors $S(p)$ and $T(p)$. It is clear that $S_0(p)$ represents the total occupation number of the plane-wave state with momentum $p$, and $S(p)$ the total spin vector associated with the particles in this state (apart from a factor). Also, $T_0(p)$ represents the amplitude of condensation of "singlet" ($S = 0$) Cooper pairs in the pair of plane wave states $(p, -p)$. The physical significance of the vector $T(p)$ is however rather less immediately obvious and we must now discuss it.
Let us represent the operator \( a_{-\beta} a_{\beta} (x, \beta = \uparrow, \downarrow) \) in the form of a matrix \( a_{-\beta} a_{\beta} \), then from Eq. (2.5) we have \( \mathbf{T} = \mathbf{T}(\mathbf{p}) \)

\[
\begin{pmatrix}
    T_y & iT_z \\
    iT_z & T_y + iT_z
\end{pmatrix}
\]

From (2.13) it is clear that the special case in which only \( S_z = \pm 1 \) pairs form ("ESP" situation) is characterized by the fact that the vector \( \mathbf{T} \) lies in the \( xy \)-plane. Restricting ourselves to this case for the moment, let us distinguish the cases (a) \( \mathbf{T} \) is a real vector apart from an overall scalar phase factor, and (b) \( \mathbf{T} \) is nontrivially complex (e.g., \( \mathbf{T} = x + iy \); this is the "nonunitary" case—see below). Let us first consider the former case, in which we may take \( \mathbf{T} \) to be a real vector without loss of generality. In this case it is evident that the direction of \( \mathbf{T} \) in the \( xy \)-plane corresponds to the phase relation between the \( S_z = \pm 1 \) components of the Cooper pair, which form in equal measure. Consequently, a rotation of \( \mathbf{T} \) around the \( z \)-axis corresponds to a change in the relative phase of the "up" and "down" components, a remark which is useful in interpreting the results of this paper (see Section 8). (This result applies also to nonunitary states.) Moreover, if we write the state

\[
\psi_e = \mathbf{c} \cdot \mathbf{T}^* | \mathbf{0} \rangle,
\]

where \( \mathbf{T}^* \) is not now restricted to lie in the \( xy \)-plane, it follows from the commutation relation (2.8) that the operator relation

\[
\mathbf{c} \cdot \mathbf{S} \psi_e = 0
\]

is satisfied. Consequently, we can say that whenever the vector \( \mathbf{T}^* \) is real apart from an overall phase factor, it may be regarded as creating a Cooper pair with total spin 1 in an eigenstate of spin projection zero with respect to the direction of \( \mathbf{T} \) (or \( \mathbf{T}^* \)). Thus, in this case the Cooper pair never has any average spin \( \langle S \rangle \) although it does have \( S^2 = 1 \).

In the opposite case (\( \mathbf{T} \) is nontrivially complex) it is easy to show that there is no axis with respect to which the Cooper pair is in an eigenstate of spin projection zero; rather, it has a total spin \( \langle S \rangle \) which is proportional [21] to \( \mathbf{T} \times \mathbf{T}^* \). It turns out that the condition for \( \mathbf{T} \) to be nontrivially complex is just the condition for the gap matrix to be "nonunitary"; we shall not exclude this possibility in what follows. It should be clearly realized, however, that the fact that the Cooper pairs formed in a nonunitary state have a net spin does not mean that the system as a whole has a net spin expectation value (or even a finite expectation value of the quantity \( S(n) \) to be introduced below); in fact, as we shall see below, the "spin of the pairs" is to some degree a conjugate variable to the total spin of the system. We also note that in our approximation the "unitariness" of the state will be a constant of the motion (see Eq. (4.15) below).
Any terms of the BCS "pairing" type in the Hamiltonian may be simply expressed in terms of the operators $T(p)$. For future reference it is useful to consider a general operator of the form

$$X = \sum_{i,j=0}^{3} f_{ij}(p, p') \sigma_{i,j} a_{p,i}^+ a_{p,j} a_{p',i} a_{p',j}$$

(2.16)

with the conditions $f_{ij} = f_{ji} = f_{i0} = 0$ if $i \neq 0$. All physically reasonable potentials should give BCS terms of this general form; for instance a Heisenberg-type exchange potential gives $f_{00} = 0, f_{ij}(p, p') = \delta_{ij} f(|p - p'|) (i, j = 1, 2, 3)$. After some algebraic manipulation using Eq. (2.5) we can rewrite Eq. (2.16) in the form

$$X = \frac{1}{2} \left( f_{00} - \sum_{i=1}^{3} f_{ii} \right) T_0(p) T_0(p') + \frac{1}{2} \left( f_{00} + \sum_{i=1}^{3} f_{ii} \right) T_0^*(p) \cdot T(p')$$

$$- \sum_{k,l=1}^{3} f_{kl} T_k^*(p) T_l(p').$$

(2.17)

(This leads directly to the well-known result that a ferromagnetic-type exchange interaction enhances the attraction for pairing in a triplet state [22, 23] but tends to suppress singlet pairing [24].) Notice particularly that a rotation-invariant form of $X$ leads only to terms of the form $T_0^*(p) T_0(p')$ and $T_0^*(p) \cdot T(p')$.

Armed with the commutation relations (2.6)-(2.10) and an expression of the type (2.17) for the pairing terms in the Hamiltonian, we could in principle carry out an RPA-type calculation of the equations of motion of the spin variables. Such a calculation would presumably be equivalent to the Green's-function treatment of Maki and Ebisawa [25]. However, such an approach would give the behaviour of the system only very near equilibrium; in addition, as we shall argue below (Sections 4 and 8) in its simplest form at least it may not correspond to the conditions which are physically realized for liquid $^3$He-A. We shall therefore adopt in this paper a different type of approach which uses a much coarser description of the system than that provided by the $S(p)$ and $T(p)$.

To obtain this simplified description we proceed as follows. For any direction on the Fermi surface specified by the unit vector $n$, we define the quantity

$$T(n) = (2\pi\hbar)^{-2} \int d^3 p' \delta(n - n') T(p') = -T(-n),$$

(2.18)

where $p' \equiv p'n'$ and the integral over the magnitude of $p'$ implied in (2.18) is to be cut off at some large limits symmetrically placed with respect to the Fermi surface. The volume of the system is taken to be unity. Because of the numerical factors occurring in its definition (the exponent $(-2)$ is not misprinted!) $T(n)$ has
the dimensions of an angular momentum, and for any \( f(n) \) we have the prescription

\[
\sum_p f(n) T(p) = \frac{2}{\hbar} \int \frac{d\Omega}{4\pi} f(n) T(n).
\]  

(2.19)

Clearly we could if necessary define an analogous quantity \( T_0(n) = T_0(-n) \). Also, for future reference we define a quantity \( S(n) \) in terms of \( S(p) \) by a prescription completely analogous to (2.18), and moreover put

\[
\bar{S}(n) = \frac{1}{2} [S(n) + S(-n)].
\]  

(2.20)

The total spin angular momentum of the system is

\[
S = \int \frac{d\Omega}{4\pi} S(n) \equiv \frac{\hbar}{2} \sum_p S(p).
\]  

(2.21)

The commutation relations of \( S \) and \( T(n) \) follow from Eqs. (2.6)-(2.10). In the commutator \([T_i(n), T_j'(n')]\) there occurs a term proportional to \( \delta_{ij} \) times the quantity

\[
Q(n) = 2 - (2/\hbar)[S_0(n) + S_0(-n)],
\]  

(2.22)

where \( S_0(n) \) is defined analogously to (2.18). The quantity \( Q(n) \) is the operator of the net displacement of the Fermi surface at the point \( n \) (i.e., the average of the displacements of the “up” and “down” Fermi surfaces, in the simplest case). It is zero in equilibrium in the absence of external fields, and from symmetry considerations is not affected by any field (such as a magnetic field) which does not couple to spin-independent quantities. (We neglect throughout the trivial complications which arise if the variation of the normal-state density of states near the Fermi surface is taken into account.) We shall therefore take \( Q(n) = 0 \) in what follows. Then the commutation relations of \( S \) and \( T(n) \) take the simple form [19]

\[
[S_i, S_j] = i\hbar \epsilon_{ijk} S_k,
\]  

(2.23)

\[
[S_i, T_j(n)] = i\hbar \epsilon_{ijk} T_k(n),
\]  

(2.24)

\[
[S_i, T_j'(n)] = i\hbar \epsilon_{ijk} T_k'(n),
\]  

(2.25)

\[
[T_i(n), T_j'(n')] = i\hbar \epsilon_{ijk} S_k(n)[\frac{1}{2}\delta(n - n') - \frac{1}{2}\delta(n + n')].
\]  

(2.26)

The kinematical relation (2.24) is crucial to the argument of this paper. It says, for example, that a definite value of the \( z \)-component of the total spin of the system is not compatible with a definite direction of the component of \( T(n) \) in the \( xy \)-plane. Since the direction of \( T \) in the \( xy \)-plane specifies the phase relation between the \( S_z = \pm 1 \) components of the Cooper pair wave function, this statement
is evidently the analogue of the "number-phase uncertainty relation" well known in the theory of superconductivity.

Equations (2.23)–(2.25) have the important consequence that \( S \) commutes with \( T'(n) \cdot T(n') \). This is a special case of the general statement that \( S \) commutes with any quantity which is invariant under rotation of the spin coordinates. We also note that \( S \) commutes with \( S \cdot T(n) \), a fact which is important when we consider the effects of "susceptibility anisotropy" (see Section 6 below).

Finally, we note that according to the definitions (2.18) and (2.21), the quantities \( S \) and \( T(n) \) (or more accurately \( T(n) \) integrated over some small but finite part of the Fermi surface) are macroscopic, and the commutators (2.23)–(2.26) are in general of order \( N^{-1} \) relative to the products \( S_i T_j \), etc. Thus, the commutation relations are needed only to obtain the equations of motion of \( S \) and \( T(n) \); thereafter we can treat these quantities as \( c \)-numbers.

3. The Equilibrium State: Broken Spin–Orbit Symmetry

If we neglect the dipole forces and set the external field equal to zero, the problem of obtaining the equilibrium state of an anisotropic superfluid is precisely the one treated, in the weak-coupling approximation, by Balian and Werthamer [17]. Within this approximation the pair Hamiltonian may be written, in our notation,

\[
\hat{H} = \hat{K} + \frac{1}{2} \left( \frac{\alpha^2}{\hbar} \right)^2 \int \frac{d\Omega}{4\pi} \int \frac{d\Omega'}{4\pi} V(n \cdot n') T'(n) \cdot T(n'),
\]

where \( \hat{K} \) is the kinetic energy and we have neglected, for simplicity, any spin-singlet pairing terms. This Hamiltonian is invariant under rotation of the spin and orbital coordinates separately, that is, in the language of Refs [4] and [5], it possesses "spin–orbit symmetry."

The equilibrium state is characterized by a finite quasiaverage (in the usual BCS sense) of the quantity \( T(n) \); we shall label this \( d(n) \). Note that this notation, though shared by much recent work on the subject, differs from that of BW who use \( d \) to characterize the gap rather than the pairing amplitude; we shall reserve for the former the notation \( \Delta(n) \), that is, \( \Delta(n) \) is (apart from a factor of \( i \)) the complex conjugate of the quantity \( d(n) \) defined by Eq. (48) of Ref. [17]. The relation between the two quantities is, in our notation,

\[
d(n : T) = \Delta(n : T) f(|\Delta(n)|^2 : T),
\]

where

\[
f(|\Delta(n)|^2 : T) = \frac{d\Delta}{d\epsilon} = \hbar \cdot \int_0^{\epsilon_{\text{F}}} \frac{\tanh(E(n, \epsilon)/2k_B T)}{2E(n, \epsilon)} d\epsilon,
\]
with $E(n, \epsilon) \equiv (\epsilon^2 + |\Delta(n)|^2)^{1/2}$, $(dn/d\epsilon)$ the density of states of both spins at the Fermi surface in the normal state, and $\epsilon_c$ a cutoff energy for the interaction. The quantity $f(|\Delta(n)|^2; T)$ is independent of $\Delta$ in the limit $T \to T_c$ and depends on it only weakly for all $T$ provided that $\epsilon_c/\Delta(0) \gg 1$. If this last condition is met, the gap equation, which in our notation reads simply

$$\Delta(n) = \text{const} \int \frac{d\Omega'}{4\pi} V(n \cdot n') \, d(n')$$

has solutions which are to a good approximation composed of spherical harmonics of order $l$, where $V$ is the most attractive spherical harmonic of $V(n \cdot n')$. Generally speaking there are a number of possible solutions for any given $l$-value, and the most stable one is the one which minimizes an appropriately defined fluctuation of $|\Delta(n)|^2$ (which is the squared energy gap for quasiparticle excitation); in particular, for $T \to T_c$ one should minimize $\langle |\Delta(n)|^4 \rangle / \langle |\Delta(n)|^2 \rangle^2$, where the angular brackets denote angular averages over the Fermi surface. This problem has been studied in detail by Barton and Moore [26]; for the special case of $P$-wave pairing it was shown in the original BW paper [17] that the best solution was one where $|\Delta(n)|^2 = \text{const}$ (the "BW state"), but it is not possible to achieve this for higher $l$.

However, it has recently become clear that the weak-coupling approximation is not likely to be applicable to the new phases of liquid $^3$He; in particular, it may be invalidated by strong "paramagnon" effects [18]. Once this is realized, we can say very little a priori about the nature of the equilibrium state; it may not even correspond to a single value of $l$, and certainly need not be unitary. Only for $l = 1$ are any substantial results known [18, 21, 27] and then only under the assumption of unitarity.

In the light of these uncertainties, we shall assume nothing about the equilibrium value of $d(n)$ beyond what follows from the Pauli principle (i.e., $d(n) = -d(-n)$) and from the invariances of the nondipolar Hamiltonian. Since the latter has spin–orbit symmetry it follows in particular that if $d(n)$ is an equilibrium solution then so is $d(\hat{R}n)$, where $\hat{R}$ is an arbitrary rotation (or inversion) matrix. In zero magnetic field we know that $(\hat{R}d)(n)$ is also a solution, but in finite field this is in general true only if $\hat{R}$ is a rotation around the axis of the field (which we shall take as $z$-axis). In particular the effect of the field is to force $d$ to lie in the $xy$-plane as far as possible (cf., Section 6 below); we shall assume that this is the only effect which the field has to order $(\mu B_0 / \Delta)^2$.

It is intuitively obvious that states with a finite value of $d(n)$ in general have the property of spontaneously broken spin–orbit symmetry (SBOS) defined in Ref. (5). This is true whether or not the Cooper pairs possess a total orbital angular momentum. For instance, if we consider an ESP state with $d(n) = \hat{z} \times n$, this corresponds to "up" and "down" spin Cooper pair amplitudes proportional
to \( n_x - in_y \) and \( n_x + in_y \), respectively; in such a state the group of “up” spins has a finite angular momentum and the “down” group an equal and opposite one, so that the total angular momentum of the system is zero. In contrast, the state proposed by Anderson and Brinkman [18] does have a finite total orbital angular momentum.

As mentioned above, any particular state described by a given \( d(n) \) is always degenerate, in the absence of the dipole forces, with the manifold of states \( d(\hat{R}n) \), where \( \hat{R} \) is an arbitrary rotation matrix. However, the dipole energy contributes terms of the form (see below)

\[
H_D \sim \int d\Omega \int d\Omega' \{ d^*(n) \cdot d(n') - 3 \hat{q} \cdot d^*(n) \hat{q} \cdot d(n') \},
\]  

(3.5)

where \( \hat{q} \) is a unit vector along \( n - n' \). These terms do not have spin–orbit symmetry and therefore in general will break the degeneracy, though possibly not completely. Since the dipole energy is extremely small compared to the characteristic energy differences between different degeneracy classes, we may safely confine ourselves to first-order perturbation theory, that is, use them to select a particular member of the degenerate class as the true equilibrium state. As we shall see below (Section 7) their qualitative effect is to force \( d(n) \) to lie as nearly perpendicular to \( n \) as is compatible with the nature of the class. It is essential to choose the equilibrium state of the system so that the dipole energy is minimized, otherwise all subsequent results are meaningless.

**Note.** There do exist a few cases in which the symmetry of the class of degenerate states is such that it cannot be broken by the dipole forces. An example is the case in which the components of \( d \) are proportional to the spherical harmonic \( Y_{20}(\theta, \phi) \) taken with respect to various axes. However, this example (which was pointed out to me by S. Takagi) seems to be pathological.

In a subsequent section we shall consider the problem of choosing the correct groundstate for various specific degeneracy classes. Before leaving this subject, however, we want to make a few miscellaneous remarks. First, we would like to introduce some terminology to help with the subsequent discussion: We consider the quantity

\[
K_{ij} = \int \frac{d\Omega}{4\pi} \{ d_i(n) d_j^*(n) + c.c. \}.
\]

(3.6)

Since \( K \) is a Hermitian matrix, it can evidently be diagonalized with real eigenvalues. It turns out that we shall often want to refer to the three special types of states for which the diagonal elements are proportional to (a) \( (1, 0, 0) \) (b) \( (1, 1, 0) \) and (c) \( (1, 1, 1) \) (i.e., \( K \) is proportional to a one-, two-, and three-dimensional projection operator respectively [27]). We shall refer to these types as “linear,”
"planar," and "spherical," respectively. In the case \( I = 1 \) an example of (a) is the state proposed in Ref. [18], of (b) the state called the ESP state by BW [17], and of (c) the BW state. Note that even for given \( I \) each of these types may in general contain more than one degeneracy class.

The second remark concerns an ambiguity in the language often used with respect to the BW \( p \)-state. BW showed, as we remarked above, that in the weak-coupling approximation the "best" \( p \)-state is characterized by \( | \Delta(n) |^2 = \text{const} \), and they wrote down explicitly the state \( d(n) = n \), which obviously conforms to this condition. This particular state corresponds to the Cooper pairs having total (orbital plus spin) angular momentum \( J = 0 \); it is isotropic in all its properties, not just in its energy spectrum. However, BW also noted that this state was degenerate with the class of states \( d = \hat{R} n \) (they neglected the dipole forces). The other members of this class do not have a well defined value of \( J \) in general, and while they still have an isotropic energy gap, there is no reason why all their properties should be isotropic. We shall see below (Section 7) that in the presence of the dipole forces the true ground state is not the \( J = 0 \) state and its NMR properties are very far from isotropic. I believe, therefore, that the word "isotropic" is best avoided in this context.

Finally, we would like to comment on BW's formula for the spin susceptibility (Eq. (49) of their paper). As they remark, this formula leads to the conclusion that any ESP state must have an anisotropic susceptibility. On the other hand, an elementary argument shows [5, Section 21] that any paramagnetic system described by a rotation-invariant Hamiltonian must always have a susceptibility which is isotropic in the zero-field limit. Indeed, it is physically obvious that if the spins are originally paired "up" and "down" the \( z \)-axis, say, and the magnetic field is applied in a transverse direction, the spins will eventually rotate the "pairing axis" to coincide with the field direction. This effect is not taken into account in BW's derivation (here, I would disagree with the final sentence of their footnote 23). Consequently, the "susceptibility" described by BW's formula [49] must be taken to be, not the true susceptibility, but the susceptibility "at constant \( T(n) \)." We return to this question in the next section.

4. ADIABATIC APPROXIMATION

In this section we shall introduce and discuss the basic physical approximation which is the foundation of all the results of this paper. Since we intend the theory to apply specifically to liquid \(^3\)He, we shall insert numerical values appropriate to this system. Furthermore, we shall assume that any magnetic fields in the problem are small enough that the magnetic energy is a small perturbation on the pairing energy, i.e., \( \mu \mathcal{H}_0 \ll \mathcal{A} \); crudely speaking, this condition is well satisfied if
\[ \mathcal{H}_0 < (1 - \frac{T}{T_c}) \text{kG}. \] (If by any chance we have to deal with a situation where there is a large magnetization \( M \) without a correspondingly large field, we define an "equivalent field" by \( \chi^{-1}M \) and apply the same restrictions to it.)

Apart from the inverse free Larmor frequency \( \omega_L^{-1} \), there are four characteristic times in our problem: the "inverse gap frequency" \( \frac{\hbar}{\Delta} \), the inverse "dipole frequency" \( \omega_0^{-1}(T) \) defined by Eq. (1.1), the quasiparticle relaxation time \( \tau \) and the nuclear spin relaxation time \( T_1 \) (or \( T_2 \)), as usually defined for NMR experiments. The last of these is of the order of seconds or minutes in the normal phase, and there is no reason to believe it would be drastically shortened by the superfluid transition (see also [2]); we may therefore take it to be effectively infinite for present purposes. To estimate the quasiparticle relaxation time \( \tau \), which gives the order of magnitude of the time in which a nonequilibrium distribution of the normal component corrects itself, we assume that in the normal phase it is comparable to the spin diffusion lifetime \( \tau_D \), take the latter from the high-pressure data of Wheatley [28] and note that the ultrasonic attenuation data [29] in \(^3\text{He}-A\) show that the relevant lifetime in that case does not increase by more than a factor of 5 or 6 down to the lowest temperatures reached in the experiment, which must have been fairly near the point B. Thus, an upper limit on \( \tau_D \) should be given by \( 2 \times 10^{-7} \text{sec.} \) except possibly very close to the lower phase boundary (for this, see below). Inserting approximate values for \( \Delta \) and the NMR shift, we can therefore make the order-of-magnitude estimates

\[
\begin{align*}
\frac{\hbar}{\Delta} &\sim 10^{-9}(1 - \frac{T}{T_c})^{-1/2} \text{sec,} \\
\omega_0^{-1} &\sim 2 \times 10^{-4}(1 - \frac{T}{T_c})^{-1/2} \text{sec,} \\
\tau &\lesssim 2 \times 10^{-7} \text{sec.} \tag{4.1}
\end{align*}
\]

It is therefore reasonable to assume that the inequalities

\[
\frac{\hbar}{\Delta} \ll \tau \ll \omega_0^{-1} \ll T_1 \tag{4.2}
\]

are satisfied over the whole region in which \(^3\text{He}-A\) is stable or metastable. Actually, the condition \( \frac{\hbar}{\Delta} \ll \tau \) is not essential to the argument; moreover, we shall see below that our approximation should work even when \( \omega_0\tau \gtrsim 1 \), provided that we then have \( \rho_n/\rho \ll 1 \). This means that our treatment should be valid for any superfluid phase of \(^4\text{He} \) right down to zero temperature (since it is very unlikely that we simultaneously encounter the conditions \( \rho_n \sim \rho \) and \( \omega_0\tau \gtrsim 1 \)).

In the light of these order-of-magnitude estimates we may develop the following picture of nuclear magnetic resonance in an anisotropic superfluid. First, suppose that there is no external magnetic field and a spin polarization \( S \) is somehow produced in the system. Were it not for the dipole forces, \( S \) would remain constant (since \( T_1 \) is effectively infinite). In the presence of the dipole forces, which depend
(among other things) on $T(n)$, the variable conjugate to $S$, the latter changes in time, and so does $T(n)$. However, because of the very small size of the dipole coupling constant, the motion is very slow—in fact, on a time-scale of $\omega_0^{-3}$, which, as we saw, is very much larger than both the inverse gap frequency $\hbar/\Delta$ and the quasiparticle relaxation time $\tau$. Consequently, we may conclude that while $S$ and $T(n)$ take (weakly) nonequilibrium values in the course of the motion, all other dynamical variables adjust adiabatically to their equilibrium values for the “instantaneous” values of $S$ and $T(n)$. Thus, for instance, the quasiparticle distribution takes at any given moment its “equilibrium” value subject to the values of $S$ and $T(n)$ at that moment.

It should be emphasized, here, that the variation of $T(n)$ with time is not an arbitrary one but of a very special type; as we shall confirm a posteriori below, for time-scales of the order of $\omega_0^{-1}T(n)$ undergoes a simple precession, which means that the system cycles through the states $d(n) = R\delta_{eq}(n)$ which were degenerate with the equilibrium state in the absence of the dipole forces (and of an external field). The crucial point to notice is that for this special type of motion of $T$, the “nondipolar” energy depends on $T$, if at all, only through the “susceptibility anisotropy” energy to be discussed below. Hence, leaving aside this term for the moment, the nondipolar energy of the state which is the equilibrium state for given $S$ can be written as some function $E(S)$ which can be obtained in terms of thermodynamic coefficients. Because this equilibrium state is, according to our hypothesis, actually attained, $E(S)$ can be taken as the effective Hamiltonian of the nondipolar forces, $S$ being now treated as an operator. We now include the dipole energy; if we restrict ourselves to the Cooper-pair terms, this can be written as a functional of $T(n)$, and we get the final effective Hamiltonian

$$\hat{H}_{eff} = E(S) + H_D(T(n)).$$  

(4.3)

Together with the commutation relations (2.23)–(2.26), this defines a soluble problem.

The approximation adopted here is very similar to the Born–Oppenheimer approximation familiar in the theory of molecules and solids. (See, for instance, (30)). In fact, if we make the correspondence $S \rightarrow$ nuclear coordinates, $T(n) \rightarrow$ nuclear momenta, we see that just as the nuclear kinetic-energy term is small ($\sim m/M$) relative to typical electronic energies, so the dipolar term is small ($\sim g_D/\Delta$, see below) relative to typical pairing energies; and hence just as for a given value of the nuclear coordinates we can assume that the electron cloud takes its equilibrium configuration, so here we can assume that for given $S$ the overall gap configuration (apart from orientation) and the quasiparticle distribution take their equilibrium values.

In the presence of a finite static magnetic field $\mathcal{H}_0$, the argument goes through similarly; the only difference is that there is now an extra contribution to the
motion of $S$ from the torque exerted by the field, and the characteristic frequency is now of the order of the larger of $\omega_0$ and $\omega_L$.

It is easy to find the explicit functional form of $H_{\text{eff}}(S, T(n))$. In the first place, if we confine ourselves to the "pair" terms and neglect "quasiparticle" renormalization (see below, Section 7) the dipole energy may be obtained from Eq. (30) of [5] (where it should be remembered that the $\sigma$'s are defined with an extra factor of $\frac{1}{2}$) and Eqs. (2.7) and (2.19) above:

$$H_D = \left( g_D / \hbar^2 \right) \int \frac{d\Omega}{4\pi} \int \frac{d\Omega'}{4\pi} \{ T'(n) \cdot T(n') - 3 \hat{q} \cdot T'(n) \hat{q} \cdot T(n') \}, \quad (4.4)$$

where

$$\hat{q} = (n - n')/|n - n'|, \quad g_D = 2\pi \gamma^2 \hbar^2 / 3. \quad (4.5)$$

For subsequent order-of-magnitude estimates it is convenient to define the quantity (we recall that the formula (4.4) is appropriate to unit volume)

$$g_D' = Ng_D = 2\pi \gamma^2 \hbar^2 / 3a^3, \quad (4.6)$$

where $N$ is the number of particles per unit volume and $a^3$ is the volume per particle. $g_D'$ is obviously of the order of the interaction between two neighboring $^3\text{He}$ nuclei; we note that it is extremely small compared to the Fermi energy

$$g_D'/\epsilon_F \sim 10^{-7}. \quad (4.7)$$

It is precisely the smallness of this ratio which justifies our approximation.

The form of $E(S)$ may be simply calculated from thermodynamic considerations: if the spin susceptibility $\chi$ is truly isotropic, then it has the form

$$E(S) = \frac{1}{2} \gamma^2 \chi^{-1} S^2 - \gamma S \cdot \mathbf{H}(t), \quad (4.8)$$

where $\mathbf{H}(t)$ is the total external field (including the r.f. field, if any). Note that since the process we are considering is an adiabatic variation of $S$, $\chi$ is $\frac{\partial^2 E}{\partial S^2}$ at constant entropy, which is just $(\partial^2 F/\partial S^2)_T$, the usual isothermal susceptibility.

If we were to take the effective Hamiltonian as simply a sum of (4.4) and (4.8), however, we should miss the effect of susceptibility anisotropy [31]: If we hold $T(n)$ fixed, then the inverse susceptibility is a tensor $\chi_{ij}^{-1}$ whose principal axes depend on $T(n)$ (see the remarks at the end of Section 3). This means that there must be terms in the effective Hamiltonian which couple $S$ and $T(n)$. We restrict ourselves to terms of lowest order in $S^2$, since higher terms on expected to be of relative order $(\mu \mathcal{H}_0/\Delta)^2 \ll 1$. Because $T$ transforms like a vector under spin rotation, (cf. Eq. (2.24)), the term of order $S^2$ must have the general form

$$E_{\text{anis}}(S : T(n)) = \text{const} \int \frac{d\Omega}{4\pi} \int \frac{d\Omega'}{4\pi} S \cdot T'(n) S \cdot T(n') f(n, n')$$

$$\equiv \frac{1}{2} \sum_{ij} \gamma^2 (A^{-1})_{ij} S_i S_j, \quad (4.9)$$
where the tensor $\Delta \chi^{-1}$ has principal axes which depend on $T(n)$. (Actually, in the weak-coupling theory at least we can write down an explicit expression for $\Delta \chi^{-1}$; see Eq. (49) of [17] and Section 6 below.)

The "dynamical" interaction between $S$ and $T(n)$ given by the "susceptibility anisotropy energy" (4.9) should be sharply distinguished from the kinematical interaction expressed by the commutation relation (2.24). We shall see that the former has an effect at all only in a surprisingly limited number of cases and even then does not change the qualitative nature of the results.

Collecting the terms given by (4.4), (4.8), and (4.9) we can finally write our effective Hamiltonian in the form

$$H_{\text{eff}}[S : T(n)] = \frac{1}{2} \gamma S \chi^{-1} S^2 - \gamma S \cdot \mathcal{H} + H_D[T(n)] + E_{\text{anis}}.$$  (4.10)

It is now very easy to derive the equations of motion of $S$ and $T(n)$. In fact let us define the quantities

$$H = -\frac{\partial H_{\text{eff}}}{\partial S}, \quad J(n) = -\frac{\partial H_{\text{eff}}}{\partial T(n)}.$$  (4.11)

Note that $H$ is not the external field, even within a factor; in fact, it vanishes identically in equilibrium. Using the commutation relations (2.23)-(2.26) and taking into account that the commutators of $S$ and $T$ may be neglected in comparison with their products, we obtain the symmetric equations

$$\frac{dS}{dt} = S \times H + \int \frac{d\Omega}{4\pi} [T(n) \times J(n)] + \text{c.c.},$$  (4.12)

$$\frac{dT(n)}{dt} = T(n) \times H + \tilde{S}(n) \times J(n).$$  (4.13)

Equations (4.12) and (4.13) should give a good description of the spin dynamics of any superfluid phase of liquid $^3$He for the frequencies of order $\omega_0$ and/or $\omega_L$, which are of primary interest in NMR experiments. For much lower frequencies (or to obtain the very-long-time behavior) we should, of course, have to incorporate relaxation mechanisms (see below). Note that to maintain consistency as well as to get a closed system of equations we must take $\tilde{S}(n)$ in (4.13) to have its equilibrium value for the given values of $S$ and $T(n)$; this equilibrium value is always directed along $S$ but is in general not equal to it, since for given values of $T$ the "equilibrium" spin polarization is not in general uniformly distributed over the Fermi surface. (To see this, we generalize Eq. (49) of BW [17] to calculate the quantity $\chi(n) = dS(n)/d\mathcal{H}_{\text{ext}}$.)

One remark should be made about (4.13): $T(n)$ is an order parameter analogous to that familiar from ordinary BCS theory, and we should therefore expect it to have the typical time-dependence $e^{i\omega t}/\omega$. Clearly, this does not appear in (4.13).
The point is that this time-dependence is common to all components of $T$, while physical quantities depend only on products of the form $T_i^* T_j$, in which the phase factor cancels out, so that it is quite legitimate to neglect it. On the other hand the variation of $T$ given by Eq. (4.13) does have a physical meaning. We may imagine, if we like, that we have got rid of the unwanted phase factor by choosing the zero of energy so that $\mu = 0$. (See also the discussion in Section 9.)

For the frequencies which are of primary physical interest we may actually simplify Eqs. (4.12) and (4.13) somewhat. In the first place, in the equation (4.12) for $S$, the contribution from susceptibility anisotropy vanishes (the contributions to the two terms on the right side cancel) and we may therefore replace $H$ by $-\gamma \mathcal{H}(t)$ and $J(n)$ by $J_D(n) = -\delta H_D/\delta T(n)$. Second, if we examine the second term on the right side of Eq. (4.13) we see that the contribution from susceptibility anisotropy vanishes, since the relevant part of $J(n)$ is always parallel to $S$, which as we have seen is parallel to $\vec{S}(n)$. (The results so far are rigorous.) We are therefore left with the dipole contribution to $J(n)$. Now, the contribution this makes to (4.13) is of order $g_D S T$, while the contribution of the first term is of order $\gamma^2 \chi^{-1} S T \sim (\epsilon_F/N) S T$. Thus, the ratio of the second term to the first is $g_D' / \epsilon_F \sim 10^{-2}$ and for most purposes we can simply drop it. However, it should be emphasized that this term is the only one which allows the orbital coordinates of the Cooper pairs to adjust themselves (unlike the first term, it does not correspond to a precession) and it is therefore essential to keep it if we want our theory to be consistent in all frequency ranges: see Section 6 below. However, the characteristic frequency associated with this “orbital” motion is of order $\omega_L (g_D' / \epsilon_F)$, which is of order 1 sec$^{-1}$ for even the largest fields considered here; while it may perhaps be relevant to the anomalous fluctuations observed [32] in $^3$He-A, it may be ignored for the purposes of ordinary NMR calculations, where the frequencies of interest are of order $\omega_0$ or $\omega_L$. Consequently, we drop this term from now on.

We may therefore rewrite Eqs. (4.12) and (4.13) in the simpler form [19]

$$\frac{dS}{dt} = -\gamma S \times \mathcal{H}(t) + R_D, \tag{4.14}$$

$$\frac{dT(n)}{dt} = T(n) \times H, \tag{4.15}$$

where

$$R_D = \int \frac{d\Omega}{4\pi} \{T(n) \times J_D(n) + c.c.\}, \quad J_D(n) = -\delta H_D/\delta T(n). \tag{4.16}$$

Equations (4.14)–(4.16) should provide a good description of the spin dynamics of any superfluid phase of liquid $^3$He except at ultralow frequencies. Since $H$ may be expressed, according to Eqs. (4.8) and (4.9), as a function of $S$, $\mathcal{H}(t)$ and
possibly $T(n)$, they form a closed set which may be solved, in principle, to obtain all those NMR properties for which field gradients are unimportant. In the next two sections we shall solve them for the important case of unsaturated c.w. resonance.

However, we must first discuss some objections which may be raised against the approximations adopted in this section.

(a) The first objection concerns our treatment of relaxation processes. We have assumed that any nonequilibrium distribution of the spin polarization, as a function either of quasiparticle energy or of position on the Fermi surface, relaxes back to its equilibrium value in a time short compared to the period of oscillation. On the other hand, we have not provided for any relaxation of $T(n)$ at all. It may be argued [33] that to allow for relaxation of $T(n)$ would seriously affect our results. Evidently, any relaxation would have to involve the normal component, and so at low enough temperatures should be negligible; however, it is not immediately obvious that it should be negligible in the range where $^3$He-A is actually stable.

I believe that it is indeed correct to neglect relaxation of $T$ throughout the whole of the physical region. To see why, let us consider for definiteness the concrete case in which the equilibrium state is “linear,” that is, the vector $T(n)$ is in the same direction for all points $n$ on the Fermi surface. Then to disturb this situation, i.e., to disorient the parallel directions of $T(n)$ for different $n$, costs an energy of the order of the pairing energy; on the other hand, to rotate $T(n)$ simultaneously for all $n$ costs only dipolar and possibly susceptibility anisotropy energy, and, as we have seen (Eq. (4.15)) it is just such a uniform rotation which occurs during magnetic resonance phenomena. Let us suppose, now, that all $T(n)$ have been uniformly rotated out of equilibrium in this way. If a collision with particles of the normal component now takes place its effect is (at most) to change $T(n)$ for a particular direction $n$, without simultaneously changing $T$ for the rest of the Fermi surface. As we saw, this process actually increases the pairing (coherence) energy, which generally speaking is an order of magnitude greater than the energy gain achieved by a correct overall orientation of $T$ (in making this estimate we bear in mind that $\mu H_0/\Delta \ll 1$.) As a result, the nonequilibrium orientation of the $T$-system is metastable as far as collisions are concerned, in much the same way as the states of superfluid flow of liquid $^4$He in an annulus (although the latter involves topological considerations which are absent here).

(Even if for some reason this metastability were to break down, the consideration that the rate of relaxation should be inversely proportional to the energy gain achieved leads to the conclusion that the relaxation time is long compared to $\omega_0^{-1}$ except possibly in rather high fields. We shall not give the details of the argument here).
We conclude, therefore, that the neglect of relaxation in our theory is entirely justified.

(b) A second objection makes essentially the opposite point: Is the relaxation of the variables which do relax (e.g. $S(n)$) always fast enough so that our approximation is valid? The relaxation takes place in times of the order of $\tau$, and at sufficiently low temperatures we certainly do not have $\omega \tau \ll 1$, since $\tau$ presumably tends to infinity as $T$ tends to zero. However, by the time the condition $\omega \tau \ll 1$ is violated we should have already $p_n/p \ll 1$; this means that the motion is determined almost entirely by the superfluid component, which may be presumed to follow the quasi-conserved variables $S$ and $T(n)$ adiabatically. (The situation is rather similar to that encountered in the theory of sound propagation in a superfluid: If the wavelength is sufficiently long, we pass smoothly as the temperature drops from the "hydrodynamic" regime to the "pure superfluid" regime, without ever encountering the "zero-sound" region). Our conclusions should therefore remain valid for all $T$. (See also the comment on the work of Maki and Ebisawa [25] in Section 8.)

One might possibly object that while the conditions $\omega_0 \tau \gtrsim 1$ and $p_n \sim p$ cannot be simultaneously met, the conditions $\omega_L \tau \gtrsim 1$ and $p_n \sim p$ certainly can at sufficiently high fields (in fact for $H_0 \sim 1kG$ at $T_c$ and at lower fields for $T < T_c$). It is very tempting to believe that any corrections arising from relaxation in this region will actually be not of order $\omega_L \tau$ but rather of order $(\omega_0/\omega_L)^2 \omega_L \tau \ll 1$ (the intuitive argument is that any effects should be of the order of the effect of the dipole forces, which is of order $(\omega_0/\omega_L)^2$ relative to that of the external field); however, we shall not investigate this point in detail here, since the main object of this paper is to explore the rich variety of NMR phenomena which occur in the limit of low fields ($\leq \omega_0$). We may note in passing that existing experiments [2] have certainly crossed the boundary $\omega_L \tau \sim 1$ and have not reported any particular change in behavior which is obviously attributable to it. (See also the result of Takagi [34] mentioned in Section 8.)

(c) There is also the question of our somewhat cavalier treatment of the orbital motion. Here, one should distinguish between the total angular momentum of the system and the angular momentum, or more generally the orbital motion, of the Cooper pairs (like $S$ and $T$, these are in some sense conjugate quantities). As regards the latter, we saw above that changes in it take place on a time-scale of the order of $[\omega_L/(g_D'c)]^{-1} \gtrsim 1$ sec, and it may therefore be reasonably regarded as held constant in ordinary NMR contexts [31]. As to the total orbital angular momentum $L$ of the system, it is actually not really an independent variable, since in the absence of an external field $\delta L = -\delta S$ by conservation of total angular momentum; if therefore in this simple case we were to add a term proportional to $L^2$ in the energy, it would just change the coefficient $\chi^{-1}$ in Eq. (4.10). However,
the coefficient of the extra term is by definition $1/2I$, where $I$ is the macroscopic moment of inertia of the system and is proportional to $N^{5/3}$ rather than $N$. Hence, the correction is entirely negligible for a macroscopic system. This conclusion should not be qualitatively affected by the presence of an external field.

(d) Finally, is it legitimate to keep the dipole forces only in the Cooper pair terms (4.4) and to throw away their other effects? The argument here is that these are the only terms where the dipole forces have a first-order effect; for instance, we can easily verify that the expectation value of the corresponding Hartree term is of order $gD^2$ and hence negligibly small (compare the argument of Ref. [5, Section 3]). Of course, we should have to keep other terms if we wanted to calculate $T_1$, but for present purposes this may be regarded as a phenomenological parameter.

To sum up, I would like to claim that Eqs. (4.12)-(4.13) constitute a good description of the spin dynamics of any possible superfluid phase of liquid $^3$He over the whole range of temperature $0 < T < T_c$ and for not too strong magnetic fields ($\mu B_0 < (1 - T/T_c)^{1/2}$ kG). (A possible though unlikely exception is the region $\omega_L \tau \gtrsim 1$.) Certainly they should be true to an excellent approximation in the low-field region where, as we shall now see, their implications are richest.

5. UNSATURATED C.W. RESONANCE: SIMPLE CASE

In this section and the next we shall solve Eqs. (4.14)-(4.15) for an important type of NMR problem, namely when the system is continuously irradiated with r.f. radiation so weak that saturation never occurs, or in other words the system stays very near equilibrium. In this section we shall neglect the effect of susceptibility anisotropy, returning to it in Section 6. This neglect is obviously justified in at least the following cases [19]:

(a) $T$ close to $T_c$. In this case the “anisotropic” component of the susceptibility is very small compared to the isotropic part, which gives the zero-order effect.

(b) Longitudinal resonance (r.f. field polarized along external field: see below). In this case it follows from the symmetry of the problem that $S$ always remains polarized along the z-axis, and therefore the only relevant susceptibility is $\chi_{zz}$, which is just the static susceptibility $\chi$ as usually defined.

(c) “Spherical” states (see Eq. (3.6) and discussion). It is highly plausible (although perhaps not one hundred percent certain) that for any physically realistic state of this type the susceptibility is isotropic even “at fixed $T(n)$.” This is certainly true for the most important member of the class, namely the BW $P$-state (see Ref. [17]).
(d) "Linear" states. These must be of the form $T(n) = af(n)$, where $a$ is independent of $n$ and can be taken real without loss of generality. It then follows that the contribution of the susceptibility anisotropy to $H$ is parallel to $a$ and hence, according to Eq. (4.15), can have no effect on the motion.

(e) The limit of high fields. This is somewhat less obvious but follows from the results of the next section.

From Eqs. (4.15)-(4.16) we find quite generally

$$\frac{dR_{Dl}}{dt} = \sum_j \Phi_{ij}(T(n)) H_j,$$

where $i, j$ denote Cartesian components and the matrix $\Phi_{ij}$ is defined by

$$\Phi_{ij}(T(n)) = (6g_D/h^2) \Re \left[ \int \frac{d\Omega}{4\pi} \frac{d\Omega'}{4\pi} \{ \hat{q} \cdot T^*(n) \hat{q} \cdot T(n') \delta_{ij} ight.
- \left. \hat{q} \cdot T^*(n) q_j T_j(n') - (\hat{q} \times T^*(n))_i (\hat{q} \times T(n'))_j \right].$$

The trace of the matrix $\Phi$ is $-6\langle H_D \rangle$ for any state. Note that in general $\Phi$ is not a symmetric matrix; see however below.

According to Eq. (4.11), $H$ is of first order in the deviation from equilibrium. Consequently, for situations close to equilibrium, we may linearize Eq. (5.2) by taking the elements of $\Phi_{ij}$ equal to their values $\Phi_{ij}^{(0)}$ in the equilibrium state. (Strictly speaking, therefore, the $\Phi_{ij}^{(0)}$ may be functions of the external field; however, we shall neglect any such dependence, since it should be very small for $\mu \mathcal{H}_0 \ll \Delta$. As emphasized in Section 3, it is essential to choose the equilibrium state so that the dipole energy is minimized. If this is done, it is easy to show that $\Phi_{ij}^{(0)}$ is a symmetric matrix; in fact, the only apparently asymmetric term in (5.2), the second, gives (apart from a term which vanishes because of the odd parity of the $T(n')$)

$$\Phi_{ij} - \Phi_{ji} \sim \epsilon_{ijk} R_{Dk},$$

and one can see (by considering the fact that in equilibrium the dipole energy must be stationary against small rotations of the system of $T(n)$) that $R = 0$ in equilibrium. It may also be shown (cf. Ref. (35)) that the $\Phi_{ij}^{(0)}$ have the following physical significance: if the system of $T(n)$ is given a small rotation specified by an infinitesimal vector $\delta \omega$ away from its equilibrium position, then the change in the dipole energy is given by

$$\delta H_D \sim \sum_{ij} \Phi_{ij}^{(0)} \delta \omega_i \delta \omega_j.$$
It therefore follows that the eigenvalues of $\Phi^{(0)}$ must be nonnegative; some of them however may be zero. We shall assume in what follows that the principal axes of $\Phi^{(0)}$ include the $z$-axis (the axis of the external field); see however the comment on the BW state in Section 9.

So far we have not made the assumption of neglecting the susceptibility anisotropy. If we now do this, we can write, using (4.11),

$$H = -\gamma^2 \chi^{-1} S' + \gamma \mathcal{H}_{r.f.}(t),$$  \hspace{1cm} (5.5)

where $S'$ is the deviation of $S$ from its equilibrium value $\gamma^{-1} \chi_0$ and $\mathcal{H}_{r.f.}$ is the oscillating r.f. field. Writing $\omega_L = \gamma \mathcal{H}_a$ and introducing the matrix

$$(\mathcal{Q}^2)_{ij} = \gamma^2 \chi^{-1} \Phi_{ij}^{(0)},$$ \hspace{1cm} (5.6)

we can combine Eqs. (4.14), (5.1), and (5.5) and write the result after Fourier transformation with respect to time in the (matrix) form

$$(\omega^2 - \mathcal{Q}^2) S' - i\omega(\omega_L \times S') = -\gamma^{-1} \chi(\mathcal{Q}^2 \mathcal{H}_{r.f.} + i\omega \omega_L \times \mathcal{H}_{r.f.}).$$ \hspace{1cm} (5.7)

In what follows we shall choose our axes to coincide with the principal axes of $\mathcal{Q}^2$ (see above) and denote the eigenvalues by $\Omega_x^2$, $\Omega_y^2$, and $\Omega_z^2$.

Let us first consider the case of "longitudinal" resonance, that is, when the r.f. field is polarized along the static external field (a highly unconventional situation). We then obtain for the corresponding dynamic susceptibility $\chi_{zz}(\omega)$ defined in the usual way

$$\chi_{zz}(\omega) = \chi \left( \frac{\Omega_z^2}{\Omega_z^2 - \omega^2} \right).$$ \hspace{1cm} (5.8)

Thus, in strong contrast to the normal situation, we obtain a well defined resonance with a frequency independent of the external field. This "longitudinal" resonance is very characteristic of superfluids with triplet pairing; we discuss its physical significance in Section 9 below.

To investigate the transverse resonance spectrum, let us first take the limit $\omega_L \to 0$. Then we see that the problem is identical to that of a two-dimensional harmonic oscillator; unless $\Omega_z^2 = \Omega_x^2$, the oscillator is anisotropic and has two nondegenerate linearly polarized normal modes. In the special case of degeneracy only, we can choose the modes to be circularly polarized.

For finite $\omega_L$ the eigenfrequencies of the "transverse" motion are

$$\omega^2 = \frac{1}{2}[(\omega_L^2 + \Omega_x^2 + \Omega_y^2) \pm \{(\omega_L^2 + \Omega_x^2 + \Omega_y^2)^2 - 4\Omega_x^2 \Omega_y^2\}^{1/2}].$$ \hspace{1cm} (5.9)

The corresponding normal modes are in general elliptically polarized; in the
degenerate case however they remain circular. In the limit \( \omega_L \to \infty \) the upper root becomes

\[
\omega^2 = \omega_L^2 + \omega_0^2(T), \quad \omega_0^2(T) \equiv \Omega_x^2(T) + \Omega_y^2(T)
\]

in agreement with Eq. (1.1), while the lower root tends to zero. In the special case of "extreme anisotropy" in which one of the eigenvalues \( \Omega_x^2, \Omega_y^2 \) is zero, this situation is preserved for arbitrary \( \omega_L \). However, in general we find two finite-frequency roots. If we introduce the quantities

\[
\delta(\omega_L, T) = \epsilon \left( \frac{\omega_0^2}{\omega_0^2 + \omega_L^2} \right)^2, \quad \epsilon = \frac{\Omega_x^2 \Omega_y^2}{(\omega_x^2 + \omega_y^2)^2} \lesssim \frac{1}{4}
\]

then we can write the frequencies in the limit of large \( \omega_L \) as

\[
\omega_+^2 = (\omega_L^2 + \omega_0^2)(1 - \delta(\omega_L, T)), \quad \omega_-^2 = (\omega_L^2 + \omega_0^2) \delta(\omega_L, T) \simeq \epsilon \omega_0^4/\omega_L^2.
\]

It is probable [19] that a finite value of \( \epsilon \) would not have been detected in the existing experiments [2]. However, it should be emphasized that if \( \epsilon \) is finite there are in any case substantial corrections to Eq. (5.9) at the temperatures far from \( T_c \) where the effect is likely to be largest (see next section).

From Eq. (5.7) we can find the dynamic susceptibility tensor by standard techniques. We obtain

\[
\chi_{xx}(\omega) = \chi \left[ 1 - \frac{\omega^2(\omega^2 - \Omega_y^2)}{D(\omega)} \right],
\]

\[
\chi_{yy}(\omega) = \chi \left[ 1 - \frac{\omega^2(\omega^2 - \Omega_x^2)}{D(\omega)} \right],
\]

\[
\chi_{xy}(\omega) = -\chi_{yx}(\omega) = \chi \left[ \frac{i \omega L \omega_0}{D(\omega)} \right],
\]

where

\[
D(\omega) \equiv (\omega^2 - \Omega_x^2)(\omega^2 - \Omega_y^2) - \omega^2 \omega_0^2.
\]

The imaginary parts, \( \chi''(\omega) \), which give the spectral weight of the absorption in each resonance, may be conveniently written in the form

\[
\chi''_{xx}(\omega) = \frac{\pi}{2} \chi \cdot \frac{\omega(\omega^2 - \Omega_y^2)}{\omega^2 - \omega_1^2} \{ \delta(\omega - \omega_2) - \delta(\omega - \omega_1) \},
\]

where \( \omega_1, \omega_2 \) are the roots given in Eq. (5.9). \( \chi''_{yy}(\omega) \) is obviously obtained from (5.16) by replacing \( \Omega_y^2 \) by \( \Omega_x^2 \).

We may easily check that the forms (5.12)–(5.14) reduce to the appropriate
static susceptibilities in the limit $\omega \to 0$, and also obey the three sum rules (see Ref. (5))

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{\chi''_{i}(\omega)}{\omega} d\omega = \chi, \quad \text{(5.18)}$$

$$\frac{2}{\pi} \int_{0}^{\infty} \omega \chi''_{i}(\omega) d\omega = (\omega L^2 + \Omega_{i}^2)\chi \quad (i = x \text{ or } y), \quad \text{(5.19)}$$

$$\frac{2}{\pi} \int_{0}^{\infty} \Re \chi_{xy}(\omega) d\omega = \omega L\chi. \quad \text{(5.20)}$$

Thus, we may consistently assume that no significant background absorption is left out of our theory.

We see that the spectral weight of the low-frequency resonance is of order $(\omega_0/\omega_L)^4$ relative to that of the high-frequency one for large $\omega_L$. Hence it is not surprising that no such resonance has so far been detected. In the special case $\Omega_{x} = 0$ (say) this mode has zero frequency and zero weight; although it would then not be directly detectable for any $\omega_L$, it will nevertheless subtract from the weight of $\chi_{xx}''$ in the main resonance according to Eq. (5.17), and thus we would expect that the "apparent" susceptibility obtained by integrating the observed resonance absorption and dividing by the resonance frequency might fall short of the value measured in static magnetization measurements. If so, we could conclude that the system had adjusted itself so that the axis corresponding to the zero eigenvalue of $\Omega^2$ lay along the r.f. field direction; however, it is not obvious that this would happen, and in the opposite case we would notice no anomaly in the integrated absorption.

To conclude this section, we note for future reference that for the degenerate case ($\Omega_{x}^2 = \Omega_{y}^2 = \omega_0^2/2$) Eq. (5.9) may be written in the simpler form

$$\omega = \frac{1}{2}(\omega_L^2 + 2\omega_0^2)^{1/2} \pm \omega_L. \quad \text{(5.21)}$$

6. **The Effect of Susceptibility Anisotropy**

As we have seen, the dynamical interaction between the total spin polarization and the quantization axes of the Cooper pairs may be expressed by a term in the Hamiltonian of the general form

$$E_{\text{anis}}(\mathbf{S}, \mathbf{T}(\mathbf{n})) = \frac{1}{2} \int \frac{d\Omega}{4\pi} \frac{d\Omega'}{4\pi} f(\mathbf{n}, \mathbf{n'}) \mathbf{S} \cdot \mathbf{T}(\mathbf{n}) \mathbf{S} \cdot \mathbf{T}(\mathbf{n}')$$

$$= \frac{1}{2}g^2 \sum_{ij} (A_{ij}^{-1})_{ij} S_i S_j, \quad \text{(6.1)}$$
where the principal axes of the tensor $\Delta \chi^{-1}$ depend on $T(n)$. The function $f(n, n')$ may depend on $|T(n)|^2$, etc., without affecting our argument; for some simple models it may be obtained explicitly (see below).

The salient point about the term (6.1) is that it commutes with $S$ and hence has no effect whatever on the spin resonance in the absence of dipole forces. However, it does affect the motion of $T(n)$; in fact, we can continue to write (4.15), and hence also (5.1), but the field $H$ is now given not by Eq. (5.5) but by

$$H = -\gamma^2 \chi^{-1} S' + \mathcal{H}_{r.t.}(t) + H', \quad (6.2)$$

where the correction term $H'$ is

$$H' = \frac{1}{2} \int \int \frac{d\Omega}{4\pi} \frac{d\Omega'}{4\pi} f(n, n') T(n) S \cdot T'(n') + c.c. \ 4\pi \ 4\pi.$$  

Evidently, $H'$ is of first order in the deviation from equilibrium. We may therefore linearize (6.3) by considering the first-order deviations of $T(n)$, $S$ and $T'(n')$ separately. Now the term in $\delta T(n) S_0 \cdot d^* (n')$ gives zero identically, because if there was a finite value of $S_0 \cdot d^*$ this would have been incorporated in the isotropic part $\chi$ of the susceptibility and hence would not enter into the expression (6.1) (recall that the equilibrium $\chi_{zz}$ is by definition $\chi$). The term in $S'$ may be combined with the first term of (6.2), so we get

$$H = -\gamma^2 \chi^{-1} S' + \gamma \mathcal{H}_{r.t.}(t) + \Delta H,$$  

$$\Delta H = -\frac{1}{2} \int \int \frac{d\Omega}{4\pi} \frac{d\Omega'}{4\pi} \{ f(n, n') S_0 \cdot T'(n') + c.c. \},$$  

where $\chi^{-1} = \chi^{-1} + \Delta \chi^{-1}$ is the tensor "susceptibility at constant $T(n)$" evaluated in the equilibrium state. Taking the time derivative of (6.5) and rearranging the triple vector product, we can write

$$\frac{d}{dt} \Delta H_i = \sum_j \chi(\Delta \chi^{-1})_{ij} (\omega_L \times H)_j = \sum_j \alpha_{ij} (\omega_L \times H)_j,$$  

where we define the matrix $\alpha = \chi \Delta \chi^{-1}$. (All quantities are now evaluated in the equilibrium state, and $\chi$ is the true (isotropic) susceptibility.) After Fourier transformation, Eq. (6.6) along with (6.4) leads to

$$H - \frac{\alpha}{i\omega} (\omega_L \times H) = -\chi^{-1} S' + \gamma \mathcal{H}_{r.t.}.$$  

On the other hand, we obtain from Eqs. (4.14) and (5.1) the result

$$\omega^2 S' - i\omega (\omega_L \times S') + \Phi^{(0)} H = -\gamma^{-1} \chi \cdot i\omega (\omega_L \times \mathcal{H}_{r.t.}).$$  

$$\omega^2 S' - i\omega (\omega_L \times S') + \Phi^{(0)} H = -\gamma^{-1} \chi \cdot i\omega (\omega_L \times \mathcal{H}_{r.t.}).$$
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This must be solved together with (6.7). Since the z-axis is obviously a principal axis of $\alpha$ as well as of $\Phi^{(0)}$, (with $\alpha_{zz} = 0$ by definition), the equations for longitudinal resonance are completely unaffected by susceptibility anisotropy, as stated above. The equations for transverse resonance may be conveniently combined and written in the form

$$(\omega^2 - Z(\omega)) S' - i\omega \omega_L \times S' = -\gamma^2 \chi\{Y(\omega) \mathcal{H}_{r,t} + i\omega \omega_L \times \mathcal{H}_{r,t}\}. \quad (6.9)$$

Here the matrices $Y(\omega), Z(\omega)$ are defined

$$Y(\omega) = \frac{\gamma^2 \Phi^{(0)}(1 - (i\omega)^{-1} \alpha \omega_L \times)}{1 - \omega^{-2} \omega_L^2 |\alpha|^2}, \quad Z(\omega) = Y(\omega)(1 + \alpha) \chi^{-1} \quad (6.10)$$

when the antisymmetric matrix $(\omega_L \times)$ is defined in the obvious way and $|\alpha|^2$ is the determinant of the $x, y$ components of $\alpha$.

It is straightforward, if tedious, to solve Eq. (6.10) for arbitrary values of the tensors $\alpha$ and $\Phi^{(0)}$. We find in general two modes only, with frequencies which reduce in the limit $\omega_L \to 0$ to the eigenvalues of $\Phi^{(0)} \chi^{-1}$. (The fact that there are not more than two modes is most easily seen by premultiplying Eq. (6.8) by $[\Phi^{(0)}]^{-1}$ and then by $(1 - (\alpha/i\omega)(\omega_L \times))$. For the case when $[\Phi^{(0)}]^{-1}$ is singular, see below). The general formulae are somewhat cumbersome and will not be given here. Instead, we note that all cases of practical interest investigated so far satisfy the conditions that $\alpha$ and $\Phi^{(0)}$ have the same set of principal axes, and that either (a) $\Phi^{(0)} \alpha$ is the null matrix, or (b) $\alpha$ and $\Phi^{(0)}$ are both proportional to the unit matrix in the $xy$-plane. The first alternative is characteristic of "linear" states, the second of "planar" ones (see Section 7). Let us discuss the two cases separately.

First, consider the case in which $\Phi^{(0)} \alpha$ is the null matrix. (In practice, this is equivalent to $[\Phi^{(0)}]^{-1}$ being singular.) Apart from the trivial case $\Phi^{(0)} \equiv 0$, this must imply $|\alpha|^2 = 0$. Hence the quantities $Z(\omega)$ and $Y(\omega)$ reduce to $\Omega^2$ and $\chi \Omega^2$ respectively (where $\Omega^2$ was defined in Eq. (5.6)), and hence Eq. (6.9) reduces identically to Eq. (5.7) of the last section. Hence in this case the susceptibility anisotropy has no effect whatsoever on the spin resonance. As we mentioned, the case $\Phi^{(0)} \alpha = 0$ is characteristic of "linear" states (i.e., those of the form $d(n) = af(n), a$ a real vector); as we saw at the beginning of Section 5, the physical reason why the susceptibility anisotropy has no effect in this case is that the field generated by it is always parallel to the vector $T$. In fact we may easily verify that in this case the quantity $S \cdot T(n)$ remains rigorously a constant of the motion, i.e., zero.

Now let us examine the second possibility, namely that $\alpha$ and $\Phi^{(0)}$ are both proportional (in the $xy$-plane) to the unit matrix. In this case we obtain the
dynamical susceptibility most simply by premultiplying Eq. (6.8) directly by the operator \((1 - (i\omega)^{-1} \alpha \omega_L \times)\). If we introduce the scalar quantity (cf. Section 5)

\[ \Omega_{xy}^2 = \gamma^2 \Phi^{(a)} / \chi \]  

(6.11)

then the dynamical susceptibilities have the form (where \(\alpha\) is now a scalar)

\[ \chi_{xx}(\omega) - \chi_{yy}(\omega) = \chi [1 - D^{-1} \{(\omega^2 - \alpha \omega_L^2 - (1 + \alpha) \Omega_{xy}^2) \cdot (\omega^2 - \alpha \Omega_{xy}^2) + \alpha(1 - \alpha) \omega^2 \omega_L^2\}], \]  

(6.12)

\[ \chi_{xy}(\omega) = -\chi_{yx}(\omega) = \chi \cdot D^{-1} \cdot i\omega \omega_L(\omega^2 - \alpha^2 \omega_L^2 - 2\alpha \Omega_{xy}^2), \]  

(6.13)

\[ D(\omega) \equiv (\omega^2 - \alpha \omega_L^2 - (1 + \alpha) \Omega_{xy}^2)^2 - \omega^2 \omega_L^2 (1 - \alpha)^2. \]  

(6.14)

It may be easily verified that Eqs. (6.12)-(6.14) reduce to (5.12)-(5.14) (with \(\Omega_x^2 = \Omega_y^2 = \Omega_{xy}^2\)) when \(\Omega_{xy}^2 \to 0\), and to the normal-state results when \(\Omega_{xy}^2 \to 0\) (even when \(\alpha\) is finite). The resonance frequencies are most simply given in the form

\[ \omega = \frac{1}{2} \left\{ (1 + \alpha) \left( \omega_L^2 + \frac{4\Omega_{xy}^2}{1 + \alpha} \right)^{1/2} \pm (1 - \alpha) \omega_L \right\}. \]  

(6.15)

In the limit \(\omega_L \to 0\) both roots tend to \((1 + \alpha) \Omega_{xy}^2 = \Phi / \chi_T \). In the limit \(\omega_L \to \infty\) we have for the upper root (\(\alpha < 1\))

\[ \omega^2 = \omega_L^2 + 2\Omega_{xy}^2, \]  

(6.16)

that is, exactly the same formula as was obtained above (Eq. (5.10)) neglecting susceptibility anisotropy. The lower root, on the other hand, does not tend to zero as previously but to \(\omega = \alpha \omega_L\). However, the weight in this resonance is very small for \(\omega_L^2 \gg \Omega_{xy}^2\).

Since the effect of susceptibility anisotropy on the “main” resonance, at least for \(\alpha \lesssim 1\), is to make the curve of \(\omega^2\) vs \(\omega_L^2\) less convex upwards, it is evident that it actually makes it more difficult to see a departure from the formula (5.10). Since the condition \(\alpha \leq 1\) is probably fulfilled for \(^3\)He-A (see below), this means that we certainly cannot use the observed [2] frequency-dependence of the NMR shift there to exclude the case of (say) a “planar” state, i.e., the conclusion of Section 5 is actually strengthened. One might, of course, ask whether the predicted second resonance (which now may lie fairly close to the main one, if \(\alpha \sim 1\)) would have been seen, but if we put numbers into formula (6.12) we find that its strength would have been small enough to make this improbable.

There is an interesting “crossover point” when \(\alpha = 1\). At this special tempera-
ture, the two resonances coincide, so that the graph of $\omega^2$ vs $\omega_L^2$ is a straight line; in fact, Eqs. (6.12) and (6.13) then reduce to

$$
\begin{align*}
\chi_{xx}(\omega) &= \chi_{yy}(\omega) = \chi \left( \frac{-(\omega_L^2 + \Omega_{xy}^2)}{\omega^2 - (\omega_L^2 + 2\Omega_{xy}^2)} \right), \\
\chi_{xy}(\omega) &= -\chi_{yx}(\omega) = \chi \left( \frac{i\omega \omega_L}{\omega^2 - (\omega_L^2 + 2\Omega_{xy}^2)} \right),
\end{align*}
$$

(6.17)

without an extra resonance at $\omega = 0$. For $\alpha > 1$ the second (weaker) resonance actually lies above the main one.

It may be verified that the expressions (6.12)–(6.14) satisfy the second and third sum rules, Eqs. (5.18) and (5.19). However, they do not satisfy the first one (Eq. (5.17)), nor does $\chi_{xx}(\omega)$ tend to the static susceptibility $\chi$ as $\omega \to 0$. In fact, for small $\omega_L$ the $\omega \to 0$ limit tends rather to the quantity $\chi_R$, which is in general less than $\chi$. Thus, the rotational invariance is apparently violated (cf. [5, Section 2]). This is entirely to be expected; although the adiabatic Hamiltonian, including the susceptibility anisotropy term, Eq. (6.1), is rotationally invariant, we must remember that by dropping the second term of the exact equation (4.13) we have in effect frozen the "orbital" motion of the pairs, thereby building in an apparent anisotropy. If we were to keep this term, a careful calculation would presumably give an extra mode with very low frequency (probably of order $\omega_L(g_D'/\epsilon_F)$ which would restore the total isotropy and rectify the sum rule (5.17). Since we are actually working on a much shorter time-scale, the "$\omega \to 0$ limit" of $\chi(\omega)$ should actually give us the "susceptibility at constant orbital motion." In the planar case this is not equal to the true static susceptibility; for low external fields the dipole forces hold $T(n)$ in the $xy$-plane, and if we then try to create a spin polarization in this plane the susceptibility anisotropy gives an apparently reduced susceptibility, namely $\chi_R$. A similar situation arises for the more general case; however, the case of a strictly linear state must be treated with a certain amount of care, since in this case there is an extra ambiguity in the prescription $\omega \to 0$. (One should in fact let $\omega$ tend to zero first and then let the deviation from linearity tend to zero; in that case the above analysis goes through.)

To use Eqs. (6.12) and (6.13) it is necessary to find the quantity $\alpha$. This may be done by using Eq. (4.9) of [17]; if we use the standard BCS weak-coupling theory and neglect Fermi-liquid corrections, we find

$$
\alpha(T) = \frac{1 - \bar{Y}(T)}{1 + \bar{Y}(T)},
$$

(6.18)

where $\bar{Y}$ is a suitably weighted angular average of the Yosida function [36]. Thus in this case the "crossover point" $\alpha = 1$ occurs at $T = 0$. However, this is an
arrest of the weak-coupling model; in particular, the value of \( \alpha \) is sensitive to the Fermi-liquid effects which enhance the normal-state susceptibility. These may be taken into account as in [37]; because the spin polarization induced at constant \( T(n) \) is not uniform (cf. the remarks in Section 4) the expression for \( \alpha \) will in general be a function of several of the dimensionless Landau parameters \( Z_i \). If for simplicity we make the assumption that for liquid \( ^3\text{He} \) only \( Z_0(\sim -3) \) is important, then \( \alpha \) is given by the expression (cf. [37])

\[
\alpha(T) = \frac{1}{1 + Z_0/4} \left( \frac{1 - \mathcal{Y}(T)}{1 + \mathcal{Y}(T)} \right).
\]  

(6.19)

Thus, the crossover point does not occur at \( T = 0 \) but at the much higher temperature where \( \mathcal{Y} \simeq 0.6 \). In the case of liquid \( ^3\text{He} \), therefore, the crossover is likely to fall within the region of existence of the A phase.

To sum up, we conclude that the dynamical coupling between \( S \) and \( T(n) \) expressed by "susceptibility anisotropy" has surprisingly little effect on the general NMR behavior, being important only for a fairly restricted class of states and then only when \( \alpha \) is appreciable (i.e., not too close to \( T_c \)). In the simplest ("planar") case the effect is to replace (5.20) by the following equation ((6.15) rewritten in terms of \( \omega_0^2 \)):

\[
\omega = \frac{1}{2} \left\{ (1 + \alpha) \left( \omega_L^2 + \frac{2\omega_0^2}{1 + \alpha} \right)^{1/2} \pm \omega_L \right\}.
\]  

(6.20)

7. Evaluation of the Characteristic Constants

As we have seen above, all the c.w. resonance characteristics of an arbitrary anisotropic Fermi superfluid may be expressed as a function of the characteristic constants \( \Omega_i^2(T) \) \((i = x, y, z)\), plus, in some cases, the susceptibility anisotropy parameter \( \alpha(T) \). We shall now see that these constants are characteristically different for different types of pairing configuration, so that the resonance spectrum should provide a very good method of "finger-printing" the configuration actually occurring.

According to Eqs. (5.2) and (5.6), the quantities \( \Omega_i^2 \) \((i = x, y, z)\) are given by

\[
\Omega_i^2 = \frac{6\gamma^2\chi^{-1}(g_d/\hbar^3)}{4\pi} \Re \left[ \int \frac{d\Omega}{4\pi} \frac{d\Omega'}{4\pi} (\hat{q} \cdot d^*(n) \hat{q} \cdot d(n')) - (\hat{q} \times d^*(n))_i (\hat{q} \times d(n'))_i \right].
\]  

(7.1)

when we have chosen the axes so that \( \hat{Q}^2 \) is diagonal. Since the evaluation of \( d(n) \)
is to be taken in the equilibrium state in the finite external field $H_0$, the quantities $\Omega_i^2$ are strictly speaking functions of $H_0$. However, we may reasonably assume that apart from its crucial role in fixing the orientation of the equilibrium state, the field affects the $\Omega_i^2$ only to order $(\mu H_0/D)^2$. We shall therefore ignore any such field-dependence. It follows immediately from (7.1) that if $d(n)$ has the “linear” form, say

$$d(n) = \hat{y}f(n)$$

then $\Omega_y^2 = 0$. The transverse resonance spectrum then has the simple form (1.1) for all $\omega_L$. If in addition $|f(n)|^2$ is symmetric with respect to interchange of the $x$ and $z$ axes, then $\Omega_z^2 = \Omega_x^2$ and the square of the longitudinal resonance frequency is numerically equal to the transverse shift $\Delta\omega^2$. The state proposed by Anderson and Brinkman [18] has both the above properties.

To proceed further it is helpful to simplify the expression (7.1) somewhat. This can be done if we make the assumption that $d(n)$ is a linear combination of spherical harmonics corresponding to a single $l$-value; this is certainly true in the limit $T \rightarrow T_c$ and should be a reasonably good approximation even at lower temperatures. Then it has been noted by Ambegaokar and Mermin [21] and demonstrated explicitly by Takagi [34] that $Q_z^2$ can be rewritten in the simpler form (ignoring for the moment the question of renormalization)

$$\Omega_i^2 = 6\gamma \chi^{-1}(g_D/h^2) \cdot 3 \Re \left[ \int \frac{dQ}{4\pi} |n \times d(n)|^2 + n \cdot d^*(n) n_i d_i(n) - |n \cdot d(n)|^2 \right]$$

$$= C_i \Omega_0^2,$$

$$\Omega_0^2 = \frac{9}{2} \gamma \chi^{-1}(g_D/h^2) \int \frac{dQ}{4\pi} |d(n)|^2.$$

The dipole energy is proportional to the negative of the quantity $\sum_{i=1}^{3} C_i \ll 1$.

As emphasized in Section 3, given a class of states which are degenerate in the absence of the dipole interaction, we must choose the member of this class which minimizes the dipole energy, i.e., maximizes $\sum_{i=1}^{3} C_i$.

We shall now consider some states which have been considered as possible candidates for the A phase of liquid $^3$He. We confine ourselves to unitary $P$-states, since these have been the most intensively investigated [18, 21, 27, 38].

(a) The first class we consider is the one obtained from the state

$$d(n) = \hat{x}n_x + \hat{y}n_y$$

by arbitrary rotations and inversions of the spin and coordinate axes separately

\footnote{The proof can be trivially rewritten to make it clear that it actually depends only on $d$ being a combination of spherical harmonics of order $l$.}
(Remember that \( d \) is a vector in spin space, \( n \) in orbital space). This class of states is called “two-dimensional” by Mermin and Stare [27] and “planar” by Varma and Werthamer [38]. According to Refs. [27] and [18] it is never the equilibrium state; it is nevertheless worth while to consider what results it would give, since for all we know a similar state may actually be stable in the \( F \)-wave case. States of this class have no total orbital angular momentum (cf. Section 3). It is evident that the external field will confine \( d(n) \) to the \( xy \)-plane, and the dipole energy is then minimized by the unique choice.

\[
d(n) = \sqrt{\frac{3}{2}} \hat{\mathbf{z}} \times n,
\]

where we normalized \( d \) to unity over the Fermi surface. The state (7.5) is called \( A_2 \) by Maki and Ebisawa [25]. Evidently it has axial symmetry, and in fact we find [19]

\[
C_x = C_y = 1/10, \quad C_z = 4/5.
\]  

This state has effects from susceptibility anisotropy, and hence Eq. (6.20) applies to it. It gives a nontrivial splitting of the transverse resonance.

(b) The second class we consider is that generated by the spin and orbital rotations and inversions of

\[
d(n) = \hat{\mathbf{x}}(n_x + in_y),
\]

This is the “axial” manifold of Varma and Werthamer [38]; it is claimed by Anderson and Brinkman [18] to contain the equilibrium state of liquid \(^3\)He when spin fluctuation effects are included. Again the external field confines \( d(n) \) to the \( xy \)-plane; the dipole energy is minimized if we choose, say,

\[
d(n) = \sqrt{\frac{3}{2}} \hat{\mathbf{y}}(n_x + in_y),
\]

but there is a residual degeneracy corresponding to simultaneous rotation of the spin and orbital coordinates around the \( z \)-axis. In this case the Cooper pairs have a finite total angular momentum, which lies in the \( xy \)-plane and is parallel to \( d \). Presumably, in the absence of any other symmetry-breaking interaction the direction of \( d \) is fixed by the container walls; however, it can clearly rotate with very little energy, and if the state (7.8) really does represent the equilibrium state of liquid \(^3\)He-A it is tempting to ascribe at least some of the characteristic fluctuations in the NMR absorption and sound attenuation in this phase to the wandering about of \( d \) [31].

The state (7.8) is the one called \( A_1 \) by Maki and Ebisawa [25]. We find

\[
C_x = C_z = 1/5, \quad C_y = 0.
\]
Thus, this state shows the linear dependence (1.1) of the squared transverse resonance frequency on $\omega_L^2$, and the intercept is numerically equal to the longitudinal resonance frequency.

(c) A class somewhat similar to (b), which has however not usually been seriously considered as a candidate for the groundstate of liquid $^3$He-A, is the class generated by

$$d(n) = \hat{x} n_z.$$  \hspace{1cm} (7.10)

This is the "polar" manifold of Varma and Werthamer; it has been shown recently by Mermin and Ambegaokar [21] that for certain forms of the strong-coupling free energy functional it may actually contain the equilibrium state. As these authors show, in this case also the dipole energy does not lift the degeneracy completely; in fact, not only may the spin coordinates be rotated around the $z$-axis, but the orbital coordinates may be rotated around the axis of $d$. Thus, for instance, both $d = y n_z$ and $d = y n_x$ are possible, as well as all intermediate forms. If an intermediate form did occur, it would mix the longitudinal and transverse resonances, since $\mathcal{Q}_L$ is then not diagonal in the $xyz$-frame. As pointed out in [21], the stability of the transverse shift in $^3$He-A is some evidence that states of the form (7.10) do not occur there. However, if by any chance a longitudinal resonance is not found, it would be tempting to ascribe this to the occurrence of the form $y n_z$, since this is the only unitary $P$-state which is predicted to give no finite-frequency longitudinal resonance.

(d) Finally, we shall consider the "isotropic" or "BW" class of states. This class is generated from the $J = 0$ state written down explicitly by BW [17]

$$d(n) = \hat{\mathbf{n}} \equiv \hat{x} n_x + \hat{y} n_y + \hat{z} n_z.$$  \hspace{1cm} (7.11)

by arbitrary rotations or inversions of the spin or coordinate axes. It has an isotropic energy gap and zero total orbital angular momentum.

It is easy to verify that the dipole energy is minimized when the rotation is through an angle of $\cos^{-1}(-1/4)$ around an arbitrary axis. Thus, at first sight the ground state is highly degenerate. However, in the presence of a finite external field there may be a small symmetry-breaking term, for the following reason: In the presence of a field $\mathbf{H}_0$ we expect that the number of $S_z = \pm 1$ pairs is unaffected to order $\mu \mathcal{H}_0/\epsilon_F$, while the number of $S_z = 0$ pairs is reduced by a factor of order $(\mu \mathcal{H}_0/\Delta)^2$ owing to the pair-breaking effect. While we have previously ignored such small factors, it now becomes essential to take them into account. The point is that a rotation of the spin coordinates about a given axis relative to the orbital coordinates tends to "correctly" correlate the spin and orbital motion (from the point of view of the dipole forces) of those pairs having $S_z = \pm 1$ along that axis, not those with $S_z = 0$. We should therefore expect the "best" axis of rotation to
be that with respect to which the $S_z = \pm 1$ pairs most outnumber the $S_z = 0$ pairs, i.e., the $z$-axis. If this is so, the “best” BW state has the form [19]

$$d_z(n) = n_z, \quad d_x = -\frac{1}{4} n_x + \left(\frac{15}{16}\right)^{1/2} n_y, \quad d_y = -\frac{1}{4} n_y - \left(\frac{15}{16}\right)^{1/2} n_z.$$  

(7.12)

For this state we easily obtain the result

$$c_x = c_y = 0, \quad c_z = \frac{1}{2}.$$  

(7.13)

Thus, there is no transverse shift in the BW state but there is a longitudinal resonance. (Strictly speaking, we should expect a transverse shift of order $(\mu H_0/\Delta)^2 \Omega_0^2$, but this is likely to be very much too small to observe in practice.)

This result may be easily understood by reference to Eq. (5.4); it costs nothing (to order $(\mu H_0/\Delta)^2$) in dipole energy to tilt the rotation axis away from the $z$-axis, but it does cost appreciable dipole energy to change the angle of rotation. It is highly plausible that a similar result applies to “spherical” states with $l > 1$.

It should be pointed out, however, that the energy determining the “correct” orientation of the BW state is a quantity of order $(\mu H_0/\Delta)^2 \langle H_D \rangle$, which is extraordinarily small by macroscopic standards. In fact if we use the estimate $\langle H_D \rangle \sim \chi \Omega_0^2/6\gamma^2$, we see that this energy is of order $(\hbar \Omega_0/\Delta)^2$ times the polarization energy of the system due to the nuclear susceptibility in the field $H_0$. Thus this energy for the whole sample is of order $10^{-13} \nu H_0^2$ ergs, where $V$ is the volume of the system measured in cm$^3$ and $H_0$ the field in gauss. A possible implication of the extraordinary weakness of this symmetry-breaking energy will be discussed in Section 9.

The four states discussed above exhaust the group of unitary $P$-states [21, 38]; we see that each of them is characterized by simple ratios of the characteristic constants $C_x, C_y, C_z$. However, we must now discuss an important objection to our evaluation of these constants for the case of real liquid $^3$He. We know that real $^3$He is a strongly interacting Fermi liquid and therefore that the excitations of the normal state at low temperatures are Landau quasiparticles rather than real particles [1]. If such a system becomes superfluid, the components of the Cooper pairs are quasiparticles [37], and hence strictly speaking the operators $S$ and $T(n)$ should be expressed in terms of quasiparticle creation and annihilation operators. On the other hand, the dipole Hamiltonian and hence the $\Omega_i^2$ are originally expressed in terms of real-particle operators, and when they are converted into quasiparticle operators we would expect the integrands occurring in Eq. (7.1) to be multiplied by some nontrivial factor. Also, the simple BCS-type decoupling implicit in Eq. (4.4) may ignore vertex renormalization (cf. [34]).

While we do not want to go into these points in detail here, it is important to be sure that they do not affect the ratios of the $\Omega_i^2$, which are so important for
where the spin fluctuation operators $\sigma_{k_\alpha}$, etc., are defined in terms of real-particle operators. Now the forces which "dress" real particles so as to convert them into quasiparticles, and which may also renormalize the spin vertices, are to an extremely good approximation spin-independent and more generally have spin-orbit symmetry (e.g. the hard-core and van der Waals forces); hence, when we reexpress $\Omega_i^2$ in terms of quasiparticles and take account of vertex renormalization, their only effect is to multiply the summand in (7.11) by some factor $R^2(k)$ which is a function only of $|k|$. Thus, their total effect on the expression (7.1) is to multiply the integrand by a factor $R^2(|n - n'|)$. (This argument may be made more explicit, and a formal expression for $R^2(|n - n'|)$ written down, by an extension of the argument of [37, Section 2 and appendix].) Now, it can be shown [39] by an extension of the argument of Takagi [34] that any such factor has the effect only of multiplying expression (7.3) by a constant, that is we get

$$\Omega_i^2 = C_i \Omega_0^2,$$

$$\Omega_0^2 = \frac{9}{2} \frac{\Omega_0}{\Omega_0^2} \gamma \chi^{-1}(g_D/h^2) \int \frac{dQ}{4\pi} |d(n)|^2, \quad (7.12)$$

where $\Omega^2$ is some suitably weighted average of $R^2$ and the quantities $C_i$ are the same as above and are quite independent of renormalization effects. The proof is given for the simple case of $P$-wave pairing in an appendix.

### TABLE I

<table>
<thead>
<tr>
<th>State</th>
<th>$C_x$</th>
<th>$C_y$</th>
<th>$C_z$</th>
<th>S.A.</th>
<th>Transverse formula</th>
<th>Long</th>
<th>$\omega_s^2/\omega_0^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) &quot;planar,&quot; $\frac{3^2}{2} \times n$</td>
<td>1/10</td>
<td>1/10</td>
<td>1/5</td>
<td>Yes</td>
<td>see (6.20)</td>
<td>Yes</td>
<td>4</td>
</tr>
<tr>
<td>(b) &quot;Axial,&quot; $\frac{3^2}{2} \hat{n}_x + in_x$</td>
<td>1/5</td>
<td>0</td>
<td>1/5</td>
<td>No</td>
<td>$\omega^2 = \omega L^2 + \omega_0^2$</td>
<td>Yes</td>
<td>1</td>
</tr>
<tr>
<td>(c) &quot;Polar&quot; $\sqrt{3} \hat{n}_x$</td>
<td>2/5</td>
<td>0</td>
<td>0</td>
<td>No</td>
<td>$\omega^2 = \omega L^2 + \omega_0^2$</td>
<td>No</td>
<td>0</td>
</tr>
<tr>
<td>(d) &quot;BW,&quot; Eq. (7.12)</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>No</td>
<td>$\omega^2 = \omega L^2$</td>
<td>Yes</td>
<td>$[\infty]</td>
</tr>
</tbody>
</table>

Let us collect the results of this section so far and tabulate them (Table I). In the first column we write down the "best" state of each class discussed above; in the case of class (c) we give the version, namely $d(n) \sim \gamma n_z$, which gives a finite transverse shift without mixing the longitudinal and transverse resonances.
The next three columns give the values of the $C_i$; the column "S.A." specifies whether or not the state shows effects of susceptibility anisotropy. The next column gives the formula for the transverse resonance frequency expressed in terms of the high-frequency shift $\omega_0^2$, and the one headed "Long" specifies whether or not there is a longitudinal resonance. Finally the last column specifies the ratio of the longitudinal resonance frequency to $\omega_0^2$.

To conclude this section, let us consider briefly the absolute magnitude and temperature-dependence of the NMR shift, that is, of the quantity $\Omega_{\alpha}^2$ (Eq. (7.12)). Taking into account the factors involved in our definition of $d(n)$ (cf. Eqs. (3.2) and (3.3)) and using standard results [6, 7, 20] of BCS-like weak-coupling theory, we find for $T$ near $T_c$

$$\int \frac{dQ}{4\pi} |d(n)|^2 \equiv \left[\frac{1}{2}(dn/d\epsilon) \ln(1 \cdot 14\epsilon/k_B T)\right]^2 \times 9 \cdot 3a(k_B T)^2 (1 - T/T_c), \quad (7.13)$$

where $a$ is the quantity $<\mathcal{A}>^2/<\mathcal{A}>^2$ and (in BCS-type theories) is the ratio of the specific heat jump to its BCS value. Collecting all the factors and putting

$$g_D = \frac{2\pi}{3} \gamma^2h^2, \quad \chi = \frac{1}{4} \cdot \gamma^2h^2(dn/d\epsilon) 1 + \frac{Z}{4}, \quad (7.14)$$

where $Z$ is the usual Landau parameter, we find [cf. [4] and erratum [4]]

$$\Omega_{\alpha}^2 \approx 28\pi\gamma^2(1 + \frac{1}{2}Z)\ln(1 \cdot 14\epsilon/k_B T)^2 \left[\ln(1 \cdot 14\epsilon/k_B T)\right]^2 (1 - T/T_c). \quad (7.15)$$

On taking the best available values of the Fermi-liquid parameters and putting $\epsilon/k_B = 0.7$ K (see [4]) we find the theoretical value

$$\Omega_{\alpha}^2/4\pi^2 \approx 2 \cdot 3 \times 10^{11} aR^2(1 - T/T_c)(Hz)^2. \quad (7.16)$$

On the other hand, the experimental value of the high-field shift, which according to the theory should be given by $(C_x + C_y) \Omega_{\alpha}^2/4\pi^2$ independent of susceptibility anisotropy (see Eqs. (5.10) and (6.16)), may be estimated [4] to be $\approx 5 \times 10^{10}$ $(1 - T/T_c)$ Hz$^2$. Since for both the "axial" and the "planar" states discussed above we have $C_x + C_y = 1/5$ (see Table I) and since the theoretical value of $a$ for both these states is 5/6, we should have to assume a value of $R$ rather larger than one to fit the data. This is not excluded (contrary to what was stated in [4]) because $R^2(|n - n'|)$ in general probably includes vertex renormalization effects, which tend to enhance the value of the spin fluctuations. However, the weak coupling model is in any case probably not adequate to describe the superfluidity of liquid $^3$He [18, 27], and in particular it is probably more realistic to take $a$ in (7.13) as given by the experimental ratio of the specific heat discontinuity to its BCS value, namely about 1.2. In that case the experimental and theoretical
values of the shift are in almost suspicious agreement if we take $\overline{R^2} = 1$, but this is probably accidental. In general the numerical uncertainties are such that all we can say at present is that the absolute magnitude (and of course the temperature-dependence) of the shift agrees satisfactorily with the theory.

8. Comparison with Other Work

In this section we comment briefly on the relation of the conclusions presented above to the earlier work of the author [4] and to the papers of Anderson [31], Maki and Ebisawa [25], and Takagi [34].

The earlier work of the author [4] obtained conclusions which are qualitatively in accordance with the present ones, but there are one or two significant differences. Apart from one or two computational errors (see erratum [4] and below) and the incorrect statement that the BW $P$-state would show a transverse shift (which was due to inadequate consideration of the rather delicate factors determining the equilibrium BW state), the differences are entirely due to the assumption made in [4] that the observed shifted resonance in $^3$He-$A$ exhausts the spectral weight. This assumption may well turn out to be true, if for instance the state of $^3$He-$A$ is the one envisaged by Anderson and Brinkman [18]; in that case, as we have seen, the result of [4] is in exact agreement with the one obtained above. However, this assumption has at the moment an approximate experimental basis only at high fields (since the integrated absorption was not measured at low fields [40]), and at high fields even a small missing component of spectral weight can affect the shift by a factor of order one. In fact, as we have seen, the high-field shift is determined not by $\Omega_a^2$ alone but by $\Omega_a^2 + \Omega_s^2$, which in the case of the "$A_2$" state considered explicitly [4] multiplies the result (Eq. (9) of that reference) by a factor of 2. Then if we take into account that the factor $F$ for the $A_2$ state, which is there quoted as 0.325, should actually be 0.075, we recover agreement with the present results. (Correction of these two errors also means that the factor of 64/13 in footnote 13 is changed to 4, in agreement with Table I.)

A treatment of the NMR shift very different from ours has been given by Anderson [31] and recently extended by Varma and Werthamer [38]. This approach uses, in principle at least, the same set of dynamical variables as ours, and makes the assumption (which we have justified explicitly in Section 4) that the orbital motion may be regarded as fixed for NMR purposes. However, the crucial equation (4.15) for the motion of $T(n)$ is replaced by the assumption that $T(n)$ comes to equilibrium for the given value of $S$ instantaneously. (Thus, $T(n)$ is implicitly treated as a classical variable which can be defined simultaneously with $S$.) To produce any shift it is then necessary to invoke the susceptibility anisotropy; as a result, no longitudinal resonance is predicted, nor any anomalous effect at all
in the BW state, and the transverse shift tends to zero with $\omega_L$ according to the formula

$$\Delta \omega^2 = \omega^2 - \omega_L^2 = \frac{\Omega_A^2}{1 + \beta \Omega_A^2 / \omega_L^2},$$

(8.1)

where the quantity $\beta$ is inversely proportional to the susceptibility anisotropy $\alpha$ defined in Section 5 above. The quantity $\beta \Omega_A^2$ should be temperature-independent to a first approximation and is probably of order $20G/\gamma$, so that it is a moot point whether the predicted deviation of $\Delta \omega^2$ from a constant at low fields would have been noticed in existing experiments. I do not believe that it is possible to reconcile the Anderson theory with the one developed here, since the basic mechanisms envisaged for the shift are completely different (see also Section 9); it is to be hoped that experiment will eventually decide between them.

A third approach to the NMR problem has been developed by Maki and Ebisawa [25] (hereafter ME) with the help of field-theoretic methods and applied specifically to the states "A₁" and "A₂" (the second and first rows of Table I). It should be noted at once that since ME do not allow their Green functions to have finite imaginary parts, their calculation corresponds to the limit $\omega \tau \gg 1$; we have argued in Section 4 that this limit does not represent the physical situation for most or all of the region in which $^3\text{He}$-A is (meta)stable, and have ourselves made the opposite assumption ($\omega \tau \ll 1$ unless $p_n \ll p$). It is therefore not necessarily surprising that some of ME's finite-temperature results differ from ours; in particular, although they find a longitudinal resonance, they predict that its frequency near $T_c$ varies as $(1 - T/T_c)^{1/4}$, rather than as $(1 - T/T_c)^{1/2}$ as predicted in Section 7 above. It should be noted that the ME formula for $\chi_{zz}(\omega)$, their Eq. (6.4), satisfies the "longitudinal" sum rule (5.18) but not the "susceptibility" sum rule (5.17); presumably, therefore, there must also be a resonance at $\omega = 0$ in their theory, which would appear if one worked at finite $k$ and then took the limit $k \to 0$. If the ME formulae should turn out to be correct near $T_c$ even though $\omega \tau \ll 1$ there, we should presumably have to conclude that the system possesses additional variables besides $S$ and $T(n)$ which are "quasiconserved"; this possibility is not allowed for in the argument of Section 4, and I do not believe it is in fact likely to occur.

In the zero-temperature limit we should expect our formulae to agree with those of ME if both are consistent, since the assumptions made in this case should be the same. However, because of the complexity of ME's formulae a detailed comparison is a nontrivial task, (in particular, ME's function $F$ (their Eq. (37)) at first sight at least appears to be singular), and we have carried it out only for the special case $\omega_L = 0$ (i.e., in ME's notation, $\omega_0 = \Omega_0 = 0$). In this case, if we also assume $\omega \ll A$, ME's function $F$ reduces simply to $(2A^2)^{-1}$. Substituting
this in their Eqs. (47) and (48), we find for the “A1” state (using their notation and interchanging the x- and y-axes to agree with our own definitions)

\[ \chi_{xx}(\omega) = \frac{-AD^2}{\omega^2 - 2AD^2(1 - I)}, \quad \chi_{yy}(\omega) = \chi_{xy}(\omega) = \chi_{yx}(\omega) = 0. \]  

(8.2)

The “A1 state” has in our language \( \Omega_y^2 = 0 \) (see Eq. (7.9)). Substituting this into the appropriate limit of our Eqs. (5.12) and (5.13), we find agreement with (8.2) (apart from an overall factor which is evidently missing in ME’s formulae) provided that we make the correspondence \( \Omega_x^2 = 2AD^2(1 - I) \). We have not checked all the factors in this equality, but obviously the general dependence on \( D^2 \) and \( \chi(\sim(1 - I)^{-1}) \) is correct.

The case of the “A2” state is a little more problematic. In general, our formulae for this case (i.e., Eqs. (6.12)–(6.14)) reduce in the limit \( \omega_L \rightarrow 0 \) to

\[ \chi_{xx}(\omega) = \chi_{yy}(\omega) = -\chi \left\{ \frac{\Omega_{xy}^2}{\omega^2 - \omega^2 - (1 + \omega) \Omega_{xy}^2} \right\}, \]  

(8.3)

\[ \chi_{xy}(\omega) = \chi_{yx}(\omega) = 0. \]

On the other hand, that of ME seems to reduce to

\[ \chi_{xx}(\omega) - \chi_{yy}(\omega) - \frac{-\frac{1}{3}AD^2}{\omega^2 - 2AD^2(1 - I)}, \]  

(8.4)

where \( A \) is said to be numerically the same parameter as in (8.2). Since \( \Omega_{xy}^2 \) for the A2 state is equal to \((1/2) \Omega_x^2 \) for the A1 state (see Table I) the results (8.3) and (8.4) agree if \( \alpha = 1 \) but not otherwise. Now \( \alpha = 1 \) at \( T = 0 \) if and only if \( I(-Z_0) = 0 \) (see the end of Section 6). I am unable to suggest a reason for this discrepancy; because of the considerations about the \( \omega \rightarrow 0 \) limit advanced in Section 6, I am inclined to regard (8.3) as more plausible than (8.4) quite independently of its derivation.

Finally, the results of ME for the longitudinal resonance at zero temperature are in complete agreement with ours. In summary, while a more detailed comparison is clearly desirable and there are one or two discrepancies, it is reassuring that the qualitative results of the field-theoretic approach at least are in agreement with our theory. We note in particular that the comparison supports the conclusion that the resonance frequencies (both longitudinal and transverse) remain finite in the limit \( \omega_L \rightarrow 0 \), in contrast to the results of [31].

Finally, I would like to comment briefly on an interesting recent preprint by Takagi [34]. He has used a perturbation expansion in (among other things) \( \omega_0^2/\omega_L^2 \) to show that at high fields the NMR shift is uniquely determined by the sum rules, although not quite in the form envisaged in [4]. (This result is quite
general and not confined to the anisotropic superfluid case.) In the case of anisotropic superfluidity, he finds quite generally that for large $\omega_L$ the resonance frequency is

$$\omega^2 = \omega_L^2 + \Omega_x^2 + \Omega_y^2.$$  

This agrees with the results (5.10) and (6.16), and in general I would regard Takagi's conclusions as entirely consistent with those of the present paper.

9. Discussion

In this paper we have developed a theory of NMR in an anisotropic Fermi superfluid which should be general enough to describe the behavior of the new phases of liquid $^3$He (if either or both are indeed of this nature) under arbitrary (physically realistic) conditions. Except when evaluating the characteristic constants, we have made no special assumptions about the actual configuration involved; indeed, one of the main uses of this theory should be to take the "fingerprints" of the new phases, since as we have seen the various types of configuration give very characteristic NMR signatures. Our theory has the advantage (which is not shared by all its rivals) that it explicitly satisfies the sum rules and gives physically correct static susceptibilities. It is, nevertheless, perhaps not easy to visualize in physical terms, and before commenting on its implications we shall try to say a little about this (cf. also [41]).

It is easy to see that the effects explored above cannot be thought of in terms of precession around any effective field. To see this, consider the fact that as $\omega_L \to 0$ the system in general behaves like a linear (in general anisotropic) oscillator (see Section 5). In fact, we shall now see that they cannot be thought of in classical terms of all, but are most closely related to the characteristically quantum-mechanical Josephson effect.

To show this, let us discuss the simple case of longitudinal resonance and focus attention on Eq. (4.15). Let us suppose in particular that we start with the system in equilibrium in zero external field, switch on an external field along the $z$-axis and study the behavior of the system for times short compared to $T_1$. For such times $H$ is just $\gamma$ times the external field $\mathscr{H}_0$ and the vector $\mathbf{T}(\mathbf{n})$ precess uniformly around the $z$-axis with frequency $\gamma \mathscr{H}_0$. However, as we noted in Section 2, a precession of $\mathbf{T}$ around the $z$-axis means that the phase of the "up-spin" order parameter $\Phi_{++}$ is changing relative to that of the "down-spin" one $\Phi_{+-}$. In fact if $\Delta \varphi(t)$ is the phase difference, we have

$$\frac{d}{dt} \Delta \varphi(t) = 2\gamma \mathscr{H}_0.$$
But this is precisely the analog of the Josephson equation for superconductors; the “up” and “down” spin bands correspond to two bulk superconductors, and the quantity $2\gamma\Delta_0$ is just the difference in twice the chemical potential in the two bands, corresponding to the quantity $2\,\text{eV}$ in the Josephson case.

One can make an even closer analogy, not just of Eq. (4.15) but of the whole resonance phenomenon, with the “number-phase” oscillations predicted to occur in (probably hypothetical) “two-band” superconductors. This phenomenon was discussed in [42], where it was shown that if the (coherent) scattering of Cooper pairs between the bands is a weak perturbation on the scattering within each band one can use (at least at $T = 0$) an adiabatic approximation which is a much simplified analog of the one used in this paper. The basic mechanism of the oscillation is that Cooper pairs tunnel from one band into another because of the interband matrix elements, and by doing so drive up the chemical potential in the second band relative to the first; this in turn by the Josephson mechanism drives the relative phase in the two bands out of equilibrium and leads to a restoring force. In the present case we have an almost exactly analogous situation; the role of the interband matrix elements is here played by the dipole forces, which are the only forces in the problem which scatter particles between the “up” and “down” bands. Since these are very weak indeed, the conditions required for the adiabatic method to be valid are extremely well satisfied. Clearly this picture should be qualitatively valid also for the transverse resonance, although in this case we must simultaneously allow for the classical precession in the external field.

It is also possible to regard the resonances explored here as collective excitations analogous to the excitons predicted to occur in ordinary BCS superconductors (compare [42]); the analogy can perhaps be most clearly seen from the formalism of Maki and Ebisawa. However, there is the important difference that in the latter case the excitons are related to unstable solutions of the gap equation corresponding to formation of Cooper pairs with, say, $l = 2$; since such solutions are usually only weakly bound relative to the normal state, the exciton frequency lies close to the top of the gap ($\omega \sim 2\Delta$). In our case the relevant unstable solution has the same $l$-value as the groundstate but a “wrong” configuration from the point of view of the dipole energy; since the latter is extremely small, the exciton lies very low down in the gap ($\omega \ll 2\Delta$).

Let us now discuss the application of the theory to the new phases of liquid $^3$He. As remarked in Section 5, the existing experiments on the NMR shift in $^3$He-A probably lie in the region of large $\omega_L$ where all states show the same behaviour (as given in Eq. (1)). At the time of writing experiments using lower fields are in progress; it is to be hoped that they will not only discriminate between our theory and conflicting ones, but (if the former is correct) also enable us to decide unambiguously the values of $\Omega_x$ and $\Omega_y$ separately, and hence to determine the general character of the configuration involved. For this purpose one should ideally use
the temperature region close to $T_c$, since as we have seen the complications due to susceptibility anisotropy are small in this region and hence Eq. (5.9) should be a good approximation whatever the nature of the configuration. Moreover the approximation used in the evaluation of the constants should be asymptotically exact as $T \to T_c$. However, since $\Omega_0^2(T)$ is small in this region it may be experimentally difficult to work too close to $T_c$, and in that case we have to be prepared to take account of susceptibility anisotropy if necessary.

For most states our theory predicts unambiguously the existence of a "longitudinal" resonance, and the observation of this (and the temperature-dependence of the resonance frequency) should be of considerable interest. If no such resonance were to be found, we should have to conclude that the state of the liquid was a "polar" one (as in the third line of Table I or something similar). The ratio of the longitudinal resonance frequency $\omega_L^2$ to the high-field transverse shift $\omega_0^2$ should enable us to determine whether the configuration could be a unitary $P$-state (see last column of Table I). As we have seen, near $T_c$ at least the numbers quoted in the table for such states are exact.

As to the B phase, at the time of writing it is still an open question whether it is a BCS-type triplet anisotropic superfluid, a BCS-type singlet (say a $d$-state) or indeed something quite different from either. If it is a BCS-type singlet then we should expect no specially anomalous NMR behavior, and in particular no longitudinal resonance. The hypothesis that it is a BCS-type triplet is compatible with the observed absence of a transverse shift as well as with the reduced $\chi$ provided that it is of the BW type (see Table I). However, in this case we should then expect a substantial longitudinal resonance. Thus the presence or absence of this resonance should provide a crucial clue to the nature of the B phase.

If the B phase really is of the BW type, this might just conceivably explain the peculiar effect noticed by Osheroff et al. [2] in which application of a heat pulse drives away the NMR signal, often for a considerable time. The point is that, as we have seen in Section 7, the energy determining the "correct" orientation of the BW state is fantastically small; it is therefore possible that application of a heat pulse would severely disorient the state, in which case the axes of $\hat{Q}^2$ become different from their equilibrium value. A particular orientation would then in general give a shifted transverse NMR signal, and if different regions of the liquid were differently disoriented the net result would be to give a whole spectrum of shifted lines, i.e., to blur out the signal and possibly to make it invisible in practice (we recall that in the existing experiments one is always working fairly close to the much larger signal from the solid $^3$He in the chamber). However, this must be regarded as very speculative; the author's personal bet at the time of writing (which may no doubt be overtaken by events!) is that the B phase is not a BW triplet.

In this paper we have not attempted to work out the NMR behavior for situa-
tions more general than unsaturated c.w. resonance. This can obviously be done using Eqs. (4.14) and (4.15), and may give further information on the nature of the configuration involved. In general we should expect that the “resonance frequency” for large-angle precession will be different from that for near-equilibrium conditions, and this may complicate both the theory and practice of pulse-type experiments.

One final question which deserves some thought is whether the validity of the general form of (say) Eq. (5.9) actually depends on the system being an anisotropic superfluid, or whether on the contrary the formula is more general and applies to any case of broken spin-orbit symmetry (cf. [5]). In this connection the result of Takagi [34] mentioned in Section 8 is suggestive. My (unsupported) guess is that at least those formulae which do not involve the susceptibility anisotropy are indeed more general.

APPENDIX

In this appendix we shall show, for the special case of pure $P$-wave pairing, that Fermi-liquid renormalization effects do not affect the ratios of the characteristic constants $O_i^2$ defined by Eq. (7.1). As noted in the text, a proof can in fact be given [39] for pairing in any pure $l$-wave state; the argument for $l = 1$ is however particularly straightforward.

We showed in Section 7 that when we express the dipole energy in terms of operators $T(n)$ defined in terms of Landau quasiparticles rather than real particles, and allow for possible vertex renormalization when making the BCS-type decoupling, the resulting expression is the same as (4.4) except that the integrand is multiplied by a factor $R^2(\|n - n'\|) \equiv R^2(n \cdot n')$, i.e.,

$$H_D = \left( \frac{g_D}{\hbar^2} \right) \int d\Omega d\Omega' \frac{R^2(n \cdot n')}{4\pi} [T^+ (n) \cdot T(n') - 3 \hat{q} \cdot T^+ (n) \hat{q} \cdot T(n')]. \quad (A.1)$$

We shall show that for any $T(n)$ whose components are $P$-wave spherical harmonics this expression may be replaced by the simpler one

$$H_D = -\frac{3}{4} \left( \frac{g_D}{\hbar^2} \right) \bar{R}^2 \int d\Omega \frac{1}{4\pi} \{|T^+ (n)|^2 - 3 |n \cdot T(n)|^2\}, \quad (A.2)$$

where $\bar{R}^2$ is a suitably weighted angular average of $R^2(n \cdot n')$. Since the $\Omega_i^2$ may be defined in terms of the double commutator of $S$ with $H_D$ (see Eq. (7.11)) and the expression for $S$ in terms of quasiparticle operators is identical to that in terms of real-particle operators [1], it follows that the $\Omega_i^2$ are also given by the single integrals (7.3) multiplied by the constant $\bar{R}^2$, as in (7.12). Hence, the $C_i$ of Eq. (7.12) are not affected.
We consider the quantities

\[ S_{\alpha\beta\gamma\delta} = \int d\Omega \frac{d\Omega'}{4\pi^2} \hat{q}_{\alpha\beta\gamma\delta} n_{\alpha\beta\gamma\delta} R^2(\mathbf{n} \cdot \mathbf{n'}) . \]  
(A.3)

Evidently, we have from symmetry considerations

\[ S_{\alpha\beta\gamma\delta} = A\delta_{\alpha\beta}\delta_{\gamma\delta} + B(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) , \]  
(A.4)

and we easily verify that

\[ \sum_{\alpha} S_{\alpha\gamma\delta} = \frac{1}{3} R_1 \delta_{\gamma\delta} , \quad \sum_{\alpha\beta} S_{\alpha\beta\gamma\delta} = \frac{1}{4}(R_0 + R_1) , \]  
(A.5)

where \( R_0 \) and \( R_1 \) are respectively the averages of \( R^2(\mathbf{n} \cdot \mathbf{n'}) \) and \( \mathbf{n} \cdot \mathbf{n'} R^2(\mathbf{n} \cdot \mathbf{n'}) \). Consequently, we have

\[ A = (1/30)(R_0 + 3R_1) , \quad B = -(1/20)(R_0 - \frac{1}{3} R_1) . \]  
(A.6)

Now we write (where summation over repeated indices is implied)

\[ T_\alpha(\mathbf{n}) = T_\alpha n_\mu . \]  
(A.7)

Inserting this into the expression (A.1) we get (apart from the constant \( g_D/\hbar^2 \))

\[ H_D = \frac{1}{3} R_1 T_\mu^\alpha T_\alpha^\mu - 3T_\mu^\alpha T_\nu^\beta S_{\alpha\beta\mu\nu} , \]  
(A.8)

and hence from (A.4) and (A.6),

\[ H_D = -(1/20)(R_0 - \frac{1}{3} R_1)(2T_\mu^\alpha T_\mu^\alpha - 3(T_\alpha^\alpha T_\mu^\alpha + T_\mu^\alpha T_\alpha^\mu)) . \]  
(A.9)

On the other hand, we easily verify that the right side of (A.2) is given by exactly the same expression as (A.9) provided that we take

\[ \overline{R^2} = R_0 - \frac{1}{3} R_1 . \]

Consequently, Eq. (A.2) is justified.

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