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Bose Bogoliubov transformations

We explore the linear-algebra aspects of using a Bogoliubov transformation to diagonalize the second-quantized Bose Hamiltonian[1]

$$\hat{H} = a_i^* h_{ij} a_j + \frac{1}{2} a_i^* \Delta_{ij} a_j^* + \frac{1}{2} a_i \Delta_{ij}^* a_j.$$

Here h is a Hermitian n-by-n matrix and Δ is a symmetric n-by-n matrix. $\hat{H}$ therefore contains $n(2n+1)$ real parameters. The a_i and a_i^* are Bose annihilation and creation operators obeying

$$[a_i, a_j^*] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad \text{etc.}$$

If $\mathbf{a} = (a_1, \ldots a_n)^T$ is a column matrix and $\mathbf{a}^\dagger = (a_1^*, \ldots a_n^*)$ the corresponding row matrix of creation operators, we can write $\hat{H}$ as either

$$\hat{H} = \frac{1}{2} \begin{bmatrix} \mathbf{a}^\dagger & \mathbf{a}^T \end{bmatrix} \begin{bmatrix} h & \Delta \\ \Delta^* & h^* \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{a}^* \end{bmatrix} - \frac{1}{2} \mathrm{tr}\,\{h\},$$

or

$$\hat{H} = \frac{1}{2} \begin{bmatrix} \mathbf{a}^T & \mathbf{a}^\dagger \end{bmatrix} \begin{bmatrix} \Delta^* & h^* \\ h & \Delta \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{a}^* \end{bmatrix} - \frac{1}{2} \mathrm{tr}\,\{h\}.$$

The matrix

$$H = \begin{bmatrix} h & \Delta \\ \Delta^* & h^* \end{bmatrix}$$

is Hermitian while

$$\tilde{H} = \begin{bmatrix} \Delta^* & h^* \\ h & \Delta \end{bmatrix}$$

is complex symmetric.

We seek a Bogoliubov transformation

$$\begin{aligned} a_i &= U_{i\alpha} b_\alpha + V_{i\alpha}^* b_\alpha^*, \\ a_i^* &= V_{i\alpha} b_\alpha + U_{i\alpha}^* b_\alpha^*, \end{aligned}$$

[1] An entry into the *analytic* aspects of the mathematics is J.Dereziński, *Bosonic quadratic Hamiltonians*, Journal of Mathematical Physics 58, 121101 (2017) 1-45; arXiv:1608.03289.

that diagonalizes $\hat{H}$ to

$$\hat{H} = \frac{1}{2}\,[\,\mathbf{b}^\dagger \quad \mathbf{b}^T\,] \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{b}^* \end{bmatrix} + c\text{-number},$$

where $\omega = \mathrm{diag}(\omega_1, \ldots, \omega_n)$.

It is seems natural to start from the eigenvalue problem

$$\begin{bmatrix} h & \Delta \\ \Delta^* & h^* \end{bmatrix} \begin{bmatrix} u_\alpha \\ v_\alpha \end{bmatrix} = \Omega_\alpha \begin{bmatrix} u_\alpha \\ v_\alpha \end{bmatrix},$$

note that if $[u_\alpha, v_\alpha]^T$ is an eigenvector of eigenvalue Ω_α then so is $[v_\alpha^*, u_\alpha^*]^T$, and then set $U_{i\alpha} = u_{i\alpha}$, $V_{i\alpha} = v_{i\alpha}$ to get a diagonalized system of oscillators with frequencies Ω_α. However *this does not work*. The problem is that in addition to diagonalizing the H matrix we also have to preserve the commutation relations. We will see that the actual energies of the system are *not* obtained from the the eigenvalues of H.

Let us write the Bogoluibov transformation as

$$\begin{bmatrix} \mathbf{a} \\ \mathbf{a}^* \end{bmatrix} = \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{b}^* \end{bmatrix},$$

where U, V are n-by-n complex matrices. The commutation relations are preserved if we impose the conditions

$$\begin{aligned} U^\dagger U - V^\dagger V &= \mathbb{I}, \\ V^\dagger U^* - U^\dagger V^* &= 0 \quad (\Leftrightarrow V^T U \quad \text{symmetric}). \end{aligned}$$

These conditions lead to

$$\begin{aligned} & \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \\ = & \begin{bmatrix} U^T & V^T \\ V^\dagger & U^\dagger \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \\ = & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

This is the defining property of the complex symplectic group, so

$$\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \in \mathrm{Sp}(2n, \mathbb{C}).$$

The symplectic property now gives

$$\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^{-1} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^T\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} U^\dagger & -V^\dagger \\ -V^T & U^T \end{bmatrix}$$

which is of the same form as the original matrix (to get second column from first: interchange rows and complex conjugate) — as is the product of two such matrices. Thus the set of complex matrices with the imposed conditions is some subgroup of $\mathrm{Sp}(2n, \mathbb{C})$. This subgroup is actually $\mathrm{Sp}(2n, \mathbb{R})$

We can expose the hidden reality of the matrices by introducing a *real structure* on our $2n$-dimensional complex vector space. A real structure is an involutive antilinear map χ that we here define to act as

$$\chi\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{y}^* \\ \mathbf{x}^* \end{bmatrix}.$$

We immediately see that

$$\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}\chi = \chi\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}, \quad \begin{bmatrix} h & \Delta \\ \Delta^* & h^* \end{bmatrix}\chi = \chi\begin{bmatrix} h & \Delta \\ \Delta^* & h^* \end{bmatrix}.$$

Vectors of the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}^* \end{bmatrix}$$

are left fixed by χ and are declared to be "real." Once we have chosen a set of $2n$ linearly independent "real" vectors $\mathbf{e}_i$ we can append the vectors $i\mathbf{e}_i$ to them to form a basis for a $4n$-dimensional vector space over the reals. Then any vector can be expanded out as

$$\mathbf{z} = \sum_{i=1}^{2n}(x_i + iy_i)\mathbf{e}_i, \quad x_i, y_i \in \mathbb{R},$$

and matrices that commute with χ do not mix x_i's with y_j's. Commuting with the real structure χ is therefore a way of stating that our symplectic matrices are secretly real. In the $\mathbf{e}_i$ basis they are explicitly real $2n$-by-$2n$ matrices.

The infinitesimal transformations are of the form

$$\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} = \mathbb{I} + \epsilon\begin{bmatrix} u & v^* \\ v & u^* \end{bmatrix} + O(\epsilon^2),$$

and the conditions on U, V require u to be skew-hermitian and v to be symmetric. Complex matrices

$$\begin{bmatrix} u & v^* \\ v & u^* \end{bmatrix}, \quad u^\dagger = -u, \quad v^T = v$$

are therefore the secretly-real generators of $\mathfrak{sp}(2n, \mathbb{R})$.

We now want to find U, V such that

$$\begin{aligned} \hat{H} &= [\mathbf{b}^\dagger, \mathbf{b}^T] \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^\dagger \begin{bmatrix} \Delta^* & h^* \\ h & \Delta \end{bmatrix} \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{b}^* \end{bmatrix} \\ &= [\mathbf{b}^\dagger, \mathbf{b}^T] \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{b}^* \end{bmatrix}. \end{aligned}$$

The equations to be obeyed by U, V are

$$\begin{aligned} hU + \Delta V &= U\Omega, \\ -\Delta^* U - h^* V &= V\Omega. \end{aligned}$$

These — together with the $U^\dagger U - V^\dagger V = \mathbb{I}$ and $V^\dagger U^* - U^\dagger V^* = 0$ symplectic orthogonality conditions — lead to the desired diagonalization.

Before we investigate whether these equations have a solution, we make the underlying group theory more explicit by the rewriting

$$\frac{1}{2} [\, \mathbf{a}^\dagger \quad \mathbf{a}^T \,] \begin{bmatrix} h & \Delta \\ \Delta^* & h^* \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{a}^* \end{bmatrix} = \frac{1}{2} [\, \mathbf{a}^T \quad \mathbf{a}^\dagger \,] \begin{bmatrix} \Delta^* & h^* \\ h & \Delta \end{bmatrix} \begin{bmatrix} \mathbf{a} \\ \mathbf{a}^* \end{bmatrix}$$

where the second matrix is complex symmetric—but secretly real as it commutes with the real structure. Now

$$-\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta^* & h^* \\ h & \Delta \end{bmatrix} = \begin{bmatrix} h & \Delta \\ -\Delta^* & -h^* \end{bmatrix}$$

anti-commutes with the real structure — but, after introducing a factor of i to compensate for the minus sign, we see that

$$i \times \begin{bmatrix} h & \Delta \\ -\Delta^* & -h^* \end{bmatrix} = \begin{bmatrix} ih & i\Delta \\ (i\Delta)^* & (ih)^* \end{bmatrix}$$

is of the form that we have identified with the generators of $\mathfrak{sp}(2n, \mathbb{R})$. Consquently a transformation

$$\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^{-1} \begin{bmatrix} h & \Delta \\ -\Delta^* & -h^* \end{bmatrix} \begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} = \begin{bmatrix} \Omega & 0 \\ 0 & -\Omega \end{bmatrix}$$

is the operation of using an $\mathrm{Sp}(2n,\mathbb{R})$ group element to conjugate ($i\times$) an element of the $\mathfrak{sp}(2n,\mathbb{R})$ Lie algebra into ($i\times$) its Cartan algebra. Now, $\mathrm{Sp}(2n,\mathbb{R})$ is non-compact and the conjugation theorem only applies to compact groups, so it may not be possible to perform this operation. If it it *is* possible we can use

$$\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^{-1} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^{T}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

to write rewrite the conjugation operation as a congruence transformation

$$\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix}^{T}\begin{bmatrix} \Delta^* & h^* \\ h & \Delta \end{bmatrix}\begin{bmatrix} U & V^* \\ V & U^* \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ \Omega & 0 \end{bmatrix},$$

that removes the Δ's, and must be equivalent to our Bogoliubov diagonalization.

To see if we can find matrices U and V that perform these operations we begin by making everything explicitly real. We do this by introducing a set of a "Majorana" Hermitian operators α_i by setting

$$\begin{aligned} a_i &= \alpha_i + i\alpha_{i+n} \\ a_i^* &= \alpha_i - i\alpha_{i+n}. \end{aligned}$$

In terms of the α_i we have

$$\hat{H} = \begin{bmatrix} \alpha_i & \alpha_{i+n} \end{bmatrix}\begin{bmatrix} \mathrm{Re}\,(h+\Delta)_{ij} & \mathrm{Im}\,(\Delta-h)_{ij} \\ \mathrm{Im}\,(\Delta+h)_{ij} & \mathrm{Re}\,(h-\Delta)_{ij} \end{bmatrix}\begin{bmatrix} \alpha_j \\ \alpha_{j+n} \end{bmatrix}$$

The hamiltonian has become a real symmetric, $2n$-by-$2n$ matrix. Such a matrix is again defined by $n(2n+1)$ real numbers, so no generality is lost or gained by considering an arbitrary real symmetric matrix.

The hermitian α_i have commutation relations

$$[\alpha_i,\alpha_j] = \frac{i}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_{ij} = \frac{i}{2}J_{ij}$$

so the commutation-relation and hermiticity-preserving transformations are

$$\begin{bmatrix} \alpha_i' \\ \alpha_{i+n}' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \alpha_i \\ \alpha_{i+n} \end{bmatrix}, \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2n,\mathbb{R}).$$

We now appeal the the following theorem:
Williamson's theorem[2]: *If M is a real, positive definite, symmetric $2n$-by-$2n$ matrix, then there there exists $S \in \mathrm{Sp}(2n, \mathbb{R})$ such that*

$$S^T M S = \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \end{bmatrix}, \quad \Omega = \mathrm{diag}(\Omega_1, \ldots, \Omega_n).$$

The Ω_i are positive, and constitute the symplectic spectrum of M.

We can use the S from Williamson to reduce our α_i Hamiltonian to

$$\hat{H} = \begin{bmatrix} \beta_i & \beta_{i+n} \end{bmatrix} \begin{bmatrix} \Omega_i \delta_{ij} & 0 \\ 0 & \Omega_i \delta_{ij} \end{bmatrix} \begin{bmatrix} \beta_j \\ \beta_{j+n} \end{bmatrix}.$$

Now define

$$\begin{aligned} b_i &= \beta_i + i\beta_{i+N}, \\ b_i^* &= \beta_i - i\beta_{i+N}, \end{aligned}$$

so

$$\begin{aligned} \beta_i &= \tfrac{1}{2}(b_i + b_i^*), \\ \beta_{i+n} &= \tfrac{1}{2i}(b_i - b_i^*). \end{aligned}$$

Then

$$\begin{aligned} \Omega_i(\beta_i^2 + \beta_{i+n}^2) &= \frac{1}{4}\Omega_i\{(b_i + b_i^*)^2 - (b_i - b_i^*)^2\} \\ &= \frac{1}{2}\Omega_i(b_i^* b_i + b_i b_i^*), \end{aligned}$$

and

$$\begin{aligned} \hat{H} &= \frac{1}{2}\begin{bmatrix} \mathbf{b}^T & \mathbf{b}^\dagger \end{bmatrix} \begin{bmatrix} 0 & \Omega \\ \Omega & 0 \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{b}^* \end{bmatrix}, \\ &= \frac{1}{2}\begin{bmatrix} \mathbf{b}^\dagger & \mathbf{b}^T \end{bmatrix} \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{b}^* \end{bmatrix}, \end{aligned}$$

as we desired.

[2] J. Williamson, Am. J. of Math. **58**, 141-163 (1936); **59**, 599 (1936); **61**, 897-911 (1936).

Proof of Williamson's theorem: As M is positive definite we can regard it as defining a real inner product $\langle \mathbf{x}, \mathbf{y} \rangle_M = \langle \mathbf{x}, \mathbf{M}\mathbf{y} \rangle$, and work in a scaled M-eigenbasis that is orthonormal with respect to this product. If, in the original basis, the symplectic form is represented by the matrix J, then we can set $K = M^{-1}J$ so that

$$\omega(\mathbf{x}, \mathbf{y}) \equiv \langle \mathbf{x}, J\mathbf{y} \rangle = \langle \mathbf{x}, K\mathbf{y} \rangle_M.$$

In the M-orthonormal basis, the operator K is represented by a skew symmetric matrix. Now we need:
Lemma: *If K is an invertible skew-symmetric $2n$-by-$2n$ matrix we can find an orthonormal set of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}_1, \ldots, \mathbf{f}_n\}$ and positive real numbers κ_i, $i = 1, \ldots, n$ such that*

$$\begin{aligned} K\mathbf{e}_i &= \kappa_i \mathbf{f}_i, \\ K\mathbf{f}_i &= -\kappa_i \mathbf{e}_i. \end{aligned}$$

We apply the lemma to the matrix K with its associated inner-product $\langle\ ,\ \rangle_M$. We then rescale the M-orthonormal $\mathbf{e}_i$, $\mathbf{f}_i$ basis to $\mathbf{e}'_i = \kappa_i^{-1/2}\mathbf{e}_i$, $\mathbf{f}'_i = \kappa^{-1/2}\mathbf{f}_i$. In this new basis the matrix K coincides with J and so the map S from the original basis to the $\mathbf{e}'_i$, $\mathbf{f}'_i$ basis is symplectic. The Gram matrix in the new basis is

$$\begin{bmatrix} \langle \mathbf{e}'_i, \mathbf{e}'_j \rangle_M & \langle \mathbf{e}'_i, \mathbf{f}'_j \rangle_M \\ \langle \mathbf{f}'_i, \mathbf{e}'_j \rangle_M & \langle \mathbf{f}'_i, \mathbf{f}'_j \rangle_M \end{bmatrix} = \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \end{bmatrix}$$

with $\Omega_i = \kappa_i^{-1}$. This Gram matrix is, of course $S^T M S$.
Proof of Lemma: The matrix K^2 is real symmetric and is therefore diagonalizable with real eigenvectors. All the eigenvalues of K^2 are strictly negative because if $K^2\mathbf{e} = \Omega\mathbf{e}$ we have

$$0 < \langle K\mathbf{e}, K\mathbf{e} \rangle = -\langle \mathbf{e}, K^2\mathbf{e} \rangle = -\Omega \langle \mathbf{e}, \mathbf{e} \rangle,$$

and $\langle \mathbf{e}, \mathbf{e} \rangle > 0$. Assume that we can find n mutually orthonormal eigenvectors $\mathbf{e}_i$ with distinct eigenvalues $-\kappa_i^2$, and set $\kappa_i \mathbf{f}_i = K\mathbf{e}_i$ Then the $\mathbf{f}_i$ is also an eigenvector of K^2 with eigenvalue $-\kappa_i^2$, and is therefore orthogonal to $\mathbf{e}_j$, $\mathbf{f}_j$ for all $j \neq i$. Furthermore, although having the same eigenvalue, the vector $\mathbf{f}_i$ is orthogonal to $\mathbf{e}_i$ as

$$\kappa_i \langle \mathbf{e}_i, \mathbf{f}_i \rangle = \langle \mathbf{e}_i, K\mathbf{e}_i \rangle = 0$$

because of the antisymmetry of K. Lastly, $\mathbf{f}_i$ is normalized because

$$\begin{aligned}\langle \mathbf{f}_i, \mathbf{f}_i \rangle &= \kappa_i^{-2} \langle K\mathbf{e}_i, K\mathbf{e}_i \rangle \\ &= -\kappa_i^{-2} \langle \mathbf{e}_i, K^2 \mathbf{e}_i \rangle \\ &= \langle \mathbf{e}_i, \mathbf{e}_i \rangle \\ &= 1.\end{aligned}$$

If K^2 does not possess n distinct eigenvalues, add a perturbation to lift the degeneracy and take limits to select appropriate $\mathbf{e}_i$. ■

Real quadratic Hamiltonian: An alternative, and perhaps more direct, view of the same mathematics comes from considering the general quadratic hamiltonian containing x's and p's. Given a classical or quantum quadratic Hamiltonian

$$\begin{aligned}\mathcal{H}[\mathbf{p}, \mathbf{x}] &= \frac{1}{2} M_{ij} p_i p_j + \frac{1}{2} V_{ij} x_i x_j + \frac{1}{2} K_{ij} (p_i x_j + x_j p_i) \\ &= \frac{1}{2} \begin{bmatrix} \mathbf{p}^T & \mathbf{x}^T \end{bmatrix} H \begin{bmatrix} \mathbf{p} \\ \mathbf{x} \end{bmatrix},\end{aligned}$$

where H is the real, positive definite, symmetric $2N$-by-$2N$ matrix

$$H = \begin{bmatrix} M & K \\ K^T & V \end{bmatrix},$$

we seek a transformation

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{x} \end{bmatrix} = S \begin{bmatrix} \mathbf{P} \\ \mathbf{X} \end{bmatrix}$$

that diagonalizes H *via* the congruence transformation $H \to S^T H S$ and preserves either the classical Poisson brackets or the quantum commutation relations. This preservation requires

$$S^T J S = J, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

so $S \in \mathrm{Sp}[2N, \mathbb{R}]$.

Following Simon, Chaturvedi, and Srinivasan[3] we observe that H being positive definite ensures that the matrices $H^{\pm 1/2}$ are well defined. We then

[3]R. Simon, S. Chaturvedi, V. Srinivasan *Congruences and Canonical Forms for a Positive Matrix: Application to the Schweinler-Wigner Extremum Principle* JMP **40** (1999) 3632-3642; arXiv:math-ph/9811003.

see that the matrix

$$\tilde{J} = H^{-1/2} J H^{-1/2}$$

is skew symmetric and hence an element of $\mathfrak{so}[2N]$. Therefore there exists an $R \in \mathrm{O}(2N)$ that conjugates $\tilde{J}$ into the Cartan algebra of $\mathfrak{so}(2n)$ — *i.e.*

$$R^T H^{-1/2} J H^{-1/2} R = \begin{bmatrix} 0 & -\Omega^{-1} \\ \Omega^{-1} & 0 \end{bmatrix},$$

with Ω^{-1} diagonal. As we are allowing $R \in \mathrm{O}[2N]$ rather than demanding $R \in \mathrm{SO}[2N]$, we can ensure that Ω^{-1} possesses strictly positive entries. This being so, we define

$$D = \begin{bmatrix} \Omega^{1/2} & 0 \\ 0 & \Omega^{1/2} \end{bmatrix},$$

so that

$$\begin{aligned} D R^T H^{-1/2} J H^{-1/2} R D &= J, \\ D R^T H^{-1/2} H H^{-1/2} R D &= D^2. \end{aligned}$$

Thus

$$S = H^{-1/2} R D \in \mathrm{Sp}[2N, \mathbb{R}],$$

and

$$S^T H S = D^2 = \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \end{bmatrix}$$

where $\Omega = \mathrm{diag}(\omega_1, \ldots \omega_N)$. We have

$$\mathcal{H}[\mathbf{P}, \mathbf{Q}] = \frac{1}{2} \begin{bmatrix} \mathbf{P}^T & \mathbf{X}^T \end{bmatrix} \begin{bmatrix} \Omega & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} \mathbf{P} \\ \mathbf{X} \end{bmatrix} = \sum_i \frac{\omega_i}{2} (P_i^2 + X_i^2),$$

which is a set of decoupled harmonic oscillators with frequency ω_i.

The eigenvalues ω_i are most easily found from those of JH which are $\pm i\omega_i$.

Ground state: The symplectic condition

$$V^\dagger U^* - U^\dagger V^* = 0$$

implies that $X = (U^\dagger)^{-1} V^\dagger$ is a symmetric matrix. With X symmetric we then have

$$\exp\left\{ \frac{1}{2} a_i^* X_{ij} a_j^* \right\} a_k \exp\left\{ -\frac{1}{2} a_i^* X_{ij} a_j^* \right\} = a_k - X_{kj} a_j^*.$$

Now the new ground state is determined by the condition that, for all k, we have

$$\begin{aligned} 0 &= (U^\dagger)^{-1}_{kj} b_j |0\rangle_b \\ &= (a_k - [(U^\dagger)^{-1} V^\dagger]_{kj} a_j^*) |0\rangle_b \\ &= \exp\left\{\frac{1}{2} a_i^* X_{ij} a_j^*\right\} a_k \exp\left\{-\frac{1}{2} a_i^* X_{ij} a_j^*\right\} |0\rangle_b \end{aligned}$$

Since $a_k |0\rangle_a = 0$ for all k is the defining condition for the old ground state, we read off that

$$|0\rangle_b = \mathcal{N} \exp\left\{\frac{1}{2} a_i^* X_{ij} a_j^*\right\} |0\rangle_a,$$

where $\mathcal{N}$ is a normalization constant that can be found from Schwinger's version[4] of MacMahon's theorem[5]

$${}_a\langle 0| \exp\left\{\frac{1}{2} a_i X_{ij}^\dagger a_j\right\} \exp\left\{\frac{1}{2} a_k^* X_{kl} a_l^*\right\} |0\rangle_a = \frac{1}{\sqrt{\det(1 - X^\dagger X)}}.$$

Louck's Bose version of the Kibble-Slepian-Mehler formula: Let us consider the n-dimensional oscillator

$$H = \frac{1}{2} \sum_{i=1}^{n} \left(-\frac{\partial^2}{\partial x_i^2} + x_i^2\right)$$

with ground state

$$\varphi_0(\mathbf{x}) = \langle \mathbf{x} | 0 \rangle = \frac{1}{\pi^{n/4}} e^{-|\mathbf{x}|^2/2}$$

and

$$a_i^\dagger = \frac{1}{\sqrt{2}}(x_i - \partial_i), \quad a_i = \frac{1}{\sqrt{2}}(x_i + \partial_i).$$

Then Louck claims and proves that for any real symmetric matrix S such that $1 + S$ is positive definite, we have

$$\langle \mathbf{x}| \exp\left\{\frac{1}{2} a_i^\dagger S_{ij} a_j^\dagger\right\} |0\rangle = \frac{\pi^{-n/4}}{\sqrt{\det(1+S)}} \exp\left\{-\frac{1}{2} \mathbf{x}^T (1-S)(1+S)^{-1} \mathbf{x}\right\}.$$

[4] J. Schwinger *On Angular Momentum*, reprinted in Lawrence Christian Biedenharn, Hendrik Van Dam, *Quantum Theory of Angular Momentum*, Academic Press, 1965.

[5] P A. MacMahon, *Combinatory analysis*, (Cambridge University Press, 1915-16), vol 1, 93-123.

Let's check the normalization. We know from the symmetric-matrix version of MacMahon that

$$\left\| \exp\left\{ \frac{1}{2} a_i^\dagger S_{ij} a_j^\dagger \right\} |0\rangle \right\|^2 = \frac{1}{\sqrt{\det(1-S^2)}},$$

while

$$\begin{aligned} \frac{1}{\pi^{n/2}} \frac{1}{\det(1+S)} \int_{\mathbb{R}^n} \exp\left\{ -\mathbf{x}^T (1-S)(1+S)^{-1} \mathbf{x} \right\} d^n x &= \frac{\det^{-1/2}[(1-S)(1+S)^{-1}]}{\det(1+S)} \\ &= \det{}^{-1/2}[(1-S)(1+S)] \\ &= \frac{1}{\sqrt{\det(1-S^2)}}. \end{aligned}$$

The normalization is correct therefore.

Squeezed states: The operators a^2, ${a^\dagger}^2$ and $a^\dagger a + \frac{1}{2}$ generate the squeezing algebra $\mathfrak{su}(1,1)$ of the group $\mathrm{SU}(1,1) \simeq \mathrm{Sp}(2,\mathbb{R})$:

$$\begin{aligned} [{a^\dagger}^2, a^2] &= -4(a^\dagger a + \tfrac{1}{2}) \\ [(a^\dagger a + \tfrac{1}{2}), a^2] &= -2a^2 \\ [(a^\dagger a + \tfrac{1}{2}), {a^\dagger}^2] &= +2{a^\dagger}^2 \end{aligned}$$

The unitary squeezing operator

$$S(z) \stackrel{\text{def}}{=} \exp\left\{ \tfrac{1}{2}(z{a^\dagger}^2 - z^* a^2) \right\},$$

with $z = |z| e^{i\theta}$, implements the symplectic transformation

$$S^\dagger(z) \begin{bmatrix} a \\ a^\dagger \end{bmatrix} S(z) = \begin{bmatrix} \cosh|z| & e^{i\theta}\sinh|z| \\ e^{-i\theta}\sinh|z| & \cosh|z| \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix}.$$

There is a faithful non-unitary representation

$$\begin{aligned} a^2 &\simeq 2i\sigma_-, \\ {a^\dagger}^2 &\simeq 2i\sigma_+, \\ (a^\dagger a + \tfrac{1}{2}) &\simeq \sigma_3, \end{aligned}$$

from which we obtain disentangling identities

$$\begin{aligned} S(z) &= \exp\left\{\tfrac{1}{2}(z{a^\dagger}^2 - z^* a^2)\right\} \\ &= \exp\left\{e^{i\theta}\tfrac{1}{2}\tanh|z|\,{a^\dagger}^2\right\}\exp\left\{-\ln\cosh|z|(a^\dagger a + \tfrac{1}{2})\right\}\exp\left\{-e^{-i\theta}\tfrac{1}{2}\tanh|z|\,a^2\right\}, \\ &= \exp\left\{-e^{-i\theta}\tfrac{1}{2}\tanh|z|\,a^2\right\}\exp\left\{+\ln\cosh|z|(a^\dagger a + \tfrac{1}{2})\right\}\exp\left\{e^{i\theta}\tfrac{1}{2}\tanh|z|\,{a^\dagger}^2\right\}. \end{aligned}$$