Michael Stone, April 2021

de Laval flow

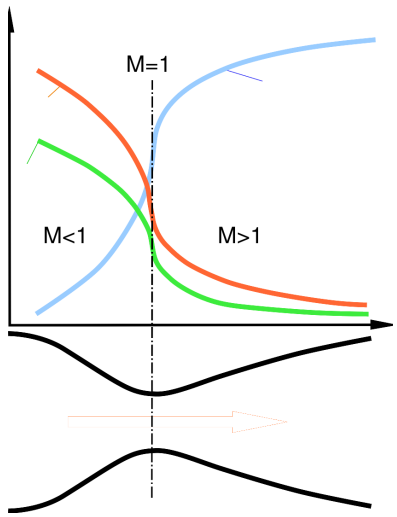

Figure 1: *A de Laval nozzle. Flow is from left to right. Red curve is temperature, green is pressure and blue is velocity. M is Mach number. (Figure from Wikipedia)*

Consider the flow of a compressible fluid through a pipe of slowly changing area $A(x)$. Here *slowly varying* means that when computing the fluid momentum we can ignore all but the x components of the velocity. The time rate of change of the x component of momentum of the fluid between two surfaces at $x_1(t)$ and $x_2(t)$ that bound a moving volume of the fluid is

$$\dot{P} = \frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho v A dx = \rho v^2 A\big|_{x_2} - \rho v^2 A\big|_{x_1} + \int_{x_1}^{x_2} \frac{\partial}{\partial t}(\rho v A) dx.$$

The x component of the total force on the same body of fluid is

$$F = pA\big|_{x_1} - pA\big|_{x_2} + \int_{x_1}^{x_2} p\frac{dA}{dx}dx,$$

where the $pA|_{x_i}$ terms are force on the body of fluid due the neighbouring fluid and the integral is the force due to the longitudinal component of pressure $p(x)$ exerted on the fluid by the wall. (When the pipe is getting wider the

unit normal to the wall has a non-zero component in the $+x$ direction.) An integration by parts allows us to rewrite the total force as

$$F = \int_{x_1}^{x_2} \left(-\frac{\partial}{\partial x}(pA) + p\frac{dA}{dx} \right) dx = \int_{x_1}^{x_2} \left(-A\frac{\partial p}{\partial x} \right) dx.$$

We can similarly write the momentum change as

$$\begin{aligned} \dot{P} &= \rho v^2 A\big|_{x_2} - \rho v^2 A\big|_{x_1} + \int_{x_1}^{x_2} \frac{\partial}{\partial t}(\rho v A) dx \\ &= \int_{x_1}^{x_2} \left(A\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho v^2 A) \right) dx. \end{aligned}$$

Now $\dot{P} = F$ and, as x_1 and x_2 are arbitrary, we can read off the local momentum conservation law

$$A\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho v^2 A) = -A\frac{\partial p}{\partial x}.$$

We also have mass conservation, so

$$\begin{aligned} 0 = \frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho A dx &= \rho v A|_{x_2} - \rho v A|_{x_1} + \int_{x_1}^{x_2} A\frac{\partial \rho}{\partial t} dx \\ &= \int_{x_1(t)}^{x_2(t)} \left(A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v A) \right) dx. \end{aligned}$$

Again, as x_1 and x_2 are arbitrary, we deduce that

$$A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v A) = 0.$$

When we subtract v times the mass conservation equation from the momentum conservation equation the derivatives of A and ρ cancel and we obtain a pipe version of Euler's equation

$$A(x)\left\{ \rho \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} \right) + \frac{\partial p}{\partial x} \right\} = 0.$$

If we write $v = \partial_x \phi(x,t)$ and observe that

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial h}{\partial x}$$

where h is the specific enthalpy $U + PV$ per unit mass, we can rewite

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

as

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + h\right) = 0$$

The statement that

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + h$$

is independent of x is Daniel Bernoulli's theorem (in a form due to Euler) for unsteady compressible flow.

For steady flow both $\partial_t v$ and $\partial_t \rho$ are zero. The mass conservation equation then becomes $\partial_x(\rho v A) = 0$, or equivalently

$$\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \frac{1}{v}\frac{\partial v}{\partial x} + \frac{1}{A}\frac{\partial A}{\partial x} = 0. \quad (\star)$$

The square of the local speed of sound is

$$c^2 = \frac{\partial p}{\partial \rho}$$

so the time independent Euler's equation can be rewritten as

$$\rho v\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial x} = -c^2\frac{\partial \rho}{\partial x} \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial \rho}{\partial x} = -\frac{v}{c^2}\frac{\partial v}{\partial x}.$$

As a consequence $(\star)$ becomes

$$\left(1 - \frac{v^2}{c^2}\right)\frac{1}{v}\frac{\partial v}{\partial x} = -\frac{1}{A}\frac{\partial A}{\partial x}.$$

This is *de Laval's equation* that says that for subsonic flow a narrowing pipe makes the fluid speed up, while for supersonic flow a *widening* pipe makes the flow speed up. This why the nozzle of a rocket engine first narrows to a throat at which the flow reaches Mach 1, and then expands allowing the exhaust gas to become supersonic.

Channel flow model: Instead of a compressible fluid in a pipe consider incompressible flow of unit mass-density fluid in a shallow channel of width

$W(x)$. The dynamical quantities are the depth $h(x,t)$ and the velocity $v(x,t)$ which we assume to be independent of y. We have the mass conservation equation

$$W\partial_t h + \partial_x(Wv) = 0,$$

and, taking into account that the pressure increases linearly with depth and a consideration of the longitudinal pressure gradient from the the walls, we have the momentum conservation equation

$$\partial_t(Whv) + \partial_x(Whv^2) = -W\partial_x\left(\frac{1}{2}gh^2\right).$$

These two equations combine to give Euler's equation

$$\partial_t v + v\partial_x v = -\partial_x(gh),$$

which makes sense as it coincides with the usual one-dimensional incompressible flow Euler equation that we expect to hold at all depths in the fluid. We can multiply this last equation by h to get

$$h(\partial_t v + v\partial_x v) = -\partial_x\left(\frac{1}{2}gh^2\right)$$

and so use channel flow to *mimic* a compressible flow in which $\rho \leftrightarrow h$ and $p \leftrightarrow (1/2)gh^2$, and hence the square of the local "speed of sound" is $c^2 = dp/d\rho \leftrightarrow gh$. Recall that $c = \sqrt{gh_0}$ is the speed of shallow water waves in a channel of undisturbed depth h_0.