Consider the flow of a compressible fluid through a pipe of slowly changing area $A(x)$. Here \textit{slowly varying} means that when computing the fluid momentum we can ignore all but the $x$ components of the velocity. The time rate of change of the $x$ component of momentum of the fluid between two surfaces at $x_1(t)$ and $x_2(t)$ that bound a moving volume of the fluid is

$$\dot{\rho v} = \frac{d}{dt} \int_{x_1(t)}^{x_2(t)} \rho v A dx = \rho v^2 A|_{x_2} - \rho v^2 A|_{x_1} + \int_{x_1}^{x_2} \frac{\partial}{\partial t} (\rho v A) dx.$$  

The $x$ component of the total force on the same body of fluid is

$$F = p A|_{x_1} - p A|_{x_2} + \int_{x_1}^{x_2} p \frac{dA}{dx} dx,$$

where the $p A|_{x_1}$ terms are force on the body of fluid due the neighbouring fluid and the integral is the force due to the longitudinal component of pressure $p(x)$ exerted on the fluid by the wall. (When the pipe is getting wider the
unit normal to the wall has a non-zero component in the +x direction.) An integration by parts allows us to rewrite the total force as

\[
F = \int_{x_1}^{x_2} \left( -\frac{\partial}{\partial x}(pA) + p\frac{dA}{dx} \right) dx = \int_{x_1}^{x_2} \left( -A\frac{\partial p}{\partial x} \right) dx.
\]

We can similarly write the momentum change as

\[
\dot{P} = \rho v^2 A|_{x_2} - \rho v^2 A|_{x_1} + \int_{x_1}^{x_2} \frac{\partial}{\partial t}(\rho v A) dx
\]

\[
= \int_{x_1}^{x_2} \left( A\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho v^2 A) \right) dx.
\]

Now \(\dot{P} = F\) and, as \(x_1\) and \(x_2\) are arbitrary, we can read off the local momentum conservation law

\[
A\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho v^2 A) = -A\frac{\partial p}{\partial x}.
\]

We also have mass conservation, so

\[
0 = \frac{d}{dt} \int_{x_1(t)}^{x_2(t)} \rho A dx = \rho v A|_{x_2} - \rho v A|_{x_1} + \int_{x_1}^{x_2} \frac{\partial}{\partial t} \rho A dx
\]

\[
= \int_{x_1(t)}^{x_2(t)} \left( A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v A) \right) dx.
\]

Again, as \(x_1\) and \(x_2\) are arbitrary, we deduce that

\[
A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v A) = 0.
\]

When we subtract \(v\) times the mass conservation equation from the momentum conservation equation the derivatives of \(A\) and \(\rho\) cancel and we obtain a pipe version of Euler’s equation

\[
A(x) \left\{ \rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + \frac{\partial p}{\partial x} \right\} = 0.
\]

If we write \(v = \partial_x \phi(x, t)\) and observe that

\[
-\frac{1}{\rho} \frac{\partial p}{\partial x} = -\frac{\partial h}{\partial x}
\]
where \( h \) is the specific enthalpy \( U + PV \) per unit mass, we can rewrite
\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}
\]
as
\[
\frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + h \right) = 0
\]
The statement that
\[
\frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + h
\]
is independent of \( x \) is Daniel Bernoulli’s theorem (in a form due to Euler) for unsteady compressible flow.

For steady flow both \( \partial_t v \) and \( \partial_t \rho \) are zero. The mass conservation equation then becomes \( \partial_x (\rho v A) = 0 \), or equivalently
\[
\frac{1}{\rho} \frac{\partial \rho}{\partial x} + \frac{1}{v} \frac{\partial v}{\partial x} + \frac{1}{A} \frac{\partial A}{\partial x} = 0. \quad (\star)
\]
The square of the local speed of sound is
\[
c^2 = \frac{\partial p}{\partial \rho}
\]
so the time independent Euler’s equation can be rewritten as
\[
\rho v \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial x} = -c^2 \frac{\partial \rho}{\partial x} \quad \Rightarrow \quad \frac{1}{\rho} \frac{\partial \rho}{\partial x} = -\frac{v}{c^2} \frac{\partial v}{\partial x}.
\]
As a consequence \( (\star) \) becomes
\[
\left( 1 - \frac{v^2}{c^2} \right) \frac{1}{v} \frac{\partial v}{\partial x} = -\frac{1}{A} \frac{\partial A}{\partial x}.
\]
This is \textit{de Laval's equation} that says that for subsonic flow a narrowing pipe makes the fluid speed up, while for supersonic flow a \textit{widening} pipe makes the flow speed up. This why the nozzle of a rocket engine first narrows to a throat at which the flow reaches Mach 1, and then expands allowing the exhaust gas to become supersonic.

**Channel flow model:** Instead of a compressible fluid in a pipe consider incompressible flow of unit mass-density fluid in a shallow channel of width...
The dynamical quantities are the depth \( h(x, t) \) and the velocity \( v(x, t) \) which we assume to be independent of \( y \). We have the mass conservation equation

\[
W \partial_t h + \partial_x(Wv) = 0,
\]

and, taking into account that the pressure increases linearly with depth and a consideration of the longitudinal pressure gradient from the the walls, we have the momentum conservation equation

\[
\partial_t(Whv) + \partial_x(Whv^2) = -W \partial_x \left( \frac{1}{2} gh^2 \right).
\]

These two equations combine to give Euler’s equation

\[
\partial_t v + v \partial_x v = -\partial_x (gh),
\]

which makes sense as it coincides with the usual one-dimensional incompressible flow Euler equation that we expect to hold at all depths in the fluid. We can multiply this last equation by \( h \) to get

\[
h(\partial_t v + v \partial_x v) = -\partial_x \left( \frac{1}{2} gh^2 \right)
\]

and so use channel flow to mimic a compressible flow in which \( \rho \leftrightarrow h \) and \( p \leftrightarrow (1/2)gh^2 \), and hence the square of the local “speed of sound” is \( c^2 = dp/d\rho \leftrightarrow gh \). Recall that \( c = \sqrt{gh_0} \) is the speed of shallow water waves in a channel of undisturbed depth \( h_0 \).