M. Stone, September 2017

Fractional charge and spectral asymmetry

Goldstone and Wilczek[1] consider a one dimensional Fermi system whose one-particle Hamiltonian is of Dirac form

$$\begin{aligned}\mathcal{H} &= -i\sigma_1\partial_x + \Delta(x)\sigma_2 e^{i\theta(x)\sigma_1} \\ &= -i\sigma_1\partial_x + \Delta(x)\sigma_2\cos\theta + \Delta(x)\sigma_3\sin\theta\end{aligned}$$

with a chirally twisted mass term. The mass Δ and chiral twist angle θ are position dependent, but asymptotically constant. G & W show that the many-fermion ground state accumulates a particle-number charge

$$Q = \int_{-\infty}^{\infty} J^0(x)\,dx = -\frac{1}{2\pi}[\theta(x)]_{-\infty}^{\infty}$$

that is localized within the region where $\theta(x)$ is changing.

Although the perturbative Feynman diagram calculations are straightforward, there are subtleties in how the charge accumulation works at the level of the eigenfunctions and eigenvalues of $\mathcal{H}$. It is therefore worthwhile to illustrate the underlying machinery by an example which is simple enough as to allow explicit computations. We will do this in the special case

$$\begin{aligned}\mathcal{H} &= -i\sigma_1\partial_x + \sigma_2 V(x) + \sigma_3 m \\ &= \begin{bmatrix} m & -i(\partial + V) \\ -i(\partial - V) & -m \end{bmatrix},\end{aligned}$$

where m is constant. One advantage of this particular model is that

$$\mathcal{H}^2 = -\partial_x^2 + m^2 + V^2 + \sigma_3\partial_x V,$$

(except for the m^2) is the standard Hamiltonian of supersymmetric quantum mechanics. If we further specialize by taking $V(x) = \tanh(x)$ then

$$\mathcal{H}^2 = \begin{bmatrix} -\partial^2 + m^2 + 1 & 0 \\ 0 & -\partial^2 + m^2 + 1 - 2\,\mathrm{sech}^2 x \end{bmatrix},$$

[1] J. Goldstone, F. Wilczek, *Fractional quantum numbers on solitons*, Phys. Rev. Lett. **47** (1981) 986-989.

contains the simplest pair of Pöschl-Teller operators. For these we have closed-form expressions for the eigenfunctions.

$\mathcal{H}^2$ has a plane-wave eigenfunction $[1,0]^T e^{ikx}$ with eigenvalue $\epsilon^2 = m^2 + 1 + k^2$. From this eigenfunction of $\mathcal{H}^2$ eigenfunctions of $\mathcal{H}$ are found by applying the projection operator decomposition

$$\mathbb{I} = \frac{1}{2\epsilon}[\epsilon + \mathcal{H}] + \frac{1}{2\epsilon}[\epsilon - \mathcal{H}]$$

to $[1,0]^T e^{ikx}$ to get

$$\begin{aligned} \psi(x) &\propto [\epsilon \pm \mathcal{H}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{ikx} \\ &= \begin{bmatrix} \epsilon \pm m \\ \mp i(ik - \tanh x) \end{bmatrix} e^{ikx}. \end{aligned}$$

In this way we obtain positive energy (u_k) and negative energy (v_k) scattering solutions

$$\begin{aligned} u_k(x) &= \frac{1}{\sqrt{2\epsilon(m+\epsilon)}} \begin{bmatrix} (m+\epsilon) \\ (k + i\tanh x) \end{bmatrix} e^{ikx}, \\ v_k(x) &= \frac{1}{\sqrt{2\epsilon(\epsilon-m)}} \begin{bmatrix} (m-\epsilon) \\ (k + i\tanh x) \end{bmatrix} e^{ikx}, \end{aligned}$$

with energies $E = \pm\epsilon$, where $\epsilon = \sqrt{m^2 + 1 + k^2}$. The normalizations have been chosen to be real and to ensure that $|u(x)|^2 = |v(x)|^2 = 1$ at large $|x|$. In particular

$$\begin{aligned} |u_k(x)|^2 &= 1 - \frac{\mathrm{sech}^2 x}{2\epsilon(\epsilon+m)}, \\ |v_k(x)|^2 &= 1 - \frac{\mathrm{sech}^2 x}{2\epsilon(\epsilon-m)}. \end{aligned}$$

Using the explicit forms of $u_k(x)$ and $v_k(x)$ we examine the scattering states' contribution

$$\begin{aligned} I(x,x') &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \{u_k(x) u_k^\dagger(x') + v_k(x) v_k^\dagger(x')\}, \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{1 + ik(\tanh x' - \tanh x) - \tanh x \tanh x'}{1+k^2} \right\} \end{aligned}$$

to the matrix-valued completeness relation. To evaluate the integral

$$K(x,x') \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} \frac{1 + ik(\tanh x' - \tanh x) - \tanh x \tanh x'}{1+k^2}$$

we need,

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} \frac{1}{1+k^2} = \frac{1}{2} e^{-|x-x'|},$$

together with its x derivative

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} \frac{ik}{1+k^2} = -\mathrm{sgn}\,(x-x') \frac{1}{2} e^{-|x-x'|}.$$

We find

$$K(x,x') = \frac{1}{2} \Big\{ 1 + \mathrm{sgn}\,(x-x')(\tanh x - \tanh x') - \tanh x \tanh x' \Big\} e^{-|x-x'|}.$$

Assuming, without loss of generality, that $x > x'$; this reduces to

$$K(x,x') = \frac{1}{2}(1+\tanh x)(1-\tanh x') e^{-(x-x')} = \frac{1}{2} \mathrm{sech}\, x\, \mathrm{sech}\, x'.$$

We thus have

$$\begin{aligned} I(x,x') &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} (u_k(x) u_k^\dagger(x') + v_k(x) v_k^\dagger(x')) \\ &= \delta(x-x') \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2} \mathrm{sech}\, x\, \mathrm{sech}\, x' \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

We see that scattering states are not quite complete, and we need to include an in-gap bound state

$$\psi_0(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{sech}\, x,$$

which has energy $E = -m$. Then we have the expected completeness relation

$$\delta(x-x') \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \psi_0(x) \psi_0^\dagger(x') + \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ u_k(x) u_k^\dagger(x') + v_k(x) v_k^\dagger(x') \right\}.$$

For any second quantized Dirac Hamiltonian with normalized eigenstates $\psi_n(x)$ and eigenvalues E_n, the ground-state has all negative energy levels

occupied. The expectation value of the normal-ordered local charge density is therefore

$$\begin{aligned}\langle \mathrm{gnd}|J^0(x)|\mathrm{gnd}\rangle &= \sum_{E_n<0} |\psi_n(x)|^2 - \frac{1}{2}\sum_{\mathrm{all}\,n} |\psi_n(x)|^2 \\ &= -\frac{1}{2}\sum \mathrm{sgn}(E_n)|\psi_n(x)|^2.\end{aligned}$$

The normal-ordering subtraction $-\frac{1}{2}\sum_{\mathrm{all}\,n} |\psi_n(x)|^2$ is an infinity $\propto \delta(0)$, but it is independent of x, the mass terms, and any potential because it results from from setting $x = x'$ in the trace of the completeness relation. The sum with $\mathrm{sgn}(E_n)$ therefore accurately tracks any *changes* in the charge density arising from adding potentials or chirally twisting the mass terms.

The scattering states' contribution to the ground-state charge density is therefore

$$\langle \mathrm{gnd}|J^0(x)|\mathrm{gnd}\rangle = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}(|u_k(x)|^2 - |v_k(x)|^2). \qquad (\star)$$

Using the explicit formulæ for $|u(x)|^2$ and $|v(x)|^2$, we find that

$$\langle \mathrm{gnd}|J^0(x)|\mathrm{gnd}\rangle = \frac{1}{4}\mathrm{sech}^2 x \left\{ \int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{1}{\epsilon(\epsilon - m)} - \int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{1}{\epsilon(\epsilon + m)} \right\}.$$

A substitution $\epsilon = \sqrt{1+m^2}\cosh s$, $k = \sqrt{1+m^2}\sinh s$ enables us to evaluate

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{1}{\epsilon(\epsilon + m)} = \frac{1}{2} - \frac{1}{\pi}\tan^{-1} m,$$

with the branch of the arc-tan lying between $-\pi/2$ and $+\pi/2$. Thus

$$\langle \mathrm{gnd}|J^0(x)|\mathrm{gnd}\rangle = \frac{1}{2\pi}(\tan^{-1} m)\,\mathrm{sech}^2 x$$

and the total chage

$$\langle Q_{\mathrm{continuum}}\rangle = \int_{-\infty}^{\infty} \langle \mathrm{gnd}|J^0(x)|\mathrm{gnd}\rangle\, dx = \frac{1}{\pi}\tan^{-1} m.$$

This is consistent with the formula of Goldstone and Wilczek. The generally non-integer charge $Q_{\mathrm{continuum}}$ is to be supplemented by $\pm 1/2$ depending on whether the $\psi_0(x)$ bound state has negative or positive energy .

A question may be raised about this calculation. In ($\star$) we seem to be assuming that for a system of large length $2L$ the density of states dn is given by $(2L)dk/2\pi$ — the same as in the free theory — and that that

$$N_+(k) \stackrel{\text{def}}{=} \int_{-L}^{L} |u_k|^2\, dx = N_-(k) \stackrel{\text{def}}{=} \int_{-L}^{L} |v_k|^2\, dx \stackrel{?}{=} 2L.$$

We know, however, from the explicit formulæ for $|u(x)|^2$ and $|v(x)|^2$ that there are $O(1)$ corrections

$$N_+(k) = 2L - \frac{1}{\epsilon(\epsilon+m)}, \quad N_-(k) = 2L - \frac{1}{\epsilon(\epsilon-m)}$$

to the normalization integral and we will see that there are also $O(1)$ corrections to the $O(L)$ positive and negative energy densities of states $dn_\pm/dk$. As we are computing an $O(1)$ total charge, should we not include these $O(1)$ corrections by replacing the measures

$$\frac{dk}{2\pi} \to N_\pm^{-1}(k)\frac{dn_\pm}{dk}\, dk$$

so as to ensure that each k mode contributes exactly unity to the total charge? That answer is "no" is shown by the manifest correctness of the completeness relation. In fact

$$\frac{dn_\pm}{dk} = \frac{1}{2\pi} N_\pm(k),$$

not just to leading order L, but also after including the $O(1)$ corrections. For Schrödinger-operator eigenfunctions we use phase shifts to show this in chapter 4 of M. Stone, P. Goldbart *Mathematics for Physics*. The same result holds for Dirac Hamiltonians[2], and for the same physical reason: far away from the scattering centre the wavefunctions cannot know of the scatterer's existence, so the x-independent normalization and density-of-states factors in the completeness relation must have a product that coincides with that of the free system.

Given a Hamiltonian H with a discrete Dirac-like spectrum E_n, its *spectral asymmetry* $\eta(H)$ is defined by a limit

$$\eta(H) = \lim_{t\to 0} \left\{ \sum_{\text{states } n} \operatorname{sgn}(E_n) e^{-t|E_n|} \right\}.$$

[2]M. Stone, *Elementary derivation of one-dimensional fermion-number fractionalization*, Physical Review B 31 (1985) 6112.

The $e^{-t|E_n|}$ cutoff is included to regulate what is often a conditionally convergent sum. For example consider $E_n = n + \alpha$ for which

$$\eta(t,\alpha) \stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} \operatorname{sgn}(\alpha+n)\exp\{-t|n+\alpha|\}.$$

The cutoff ensures that $\eta(t,\alpha) = \eta(t,\alpha+m)$ for any $m \in \mathbb{Z}$. For $0 < \alpha < 1$ we can evaluate the sum in closed form as

$$\eta(t,\alpha) = \frac{\sinh[(1-2\alpha)t/2]}{\sinh t/2}.$$

The $t \to 0$ limit is $(1-2\alpha)$. Taking into account the periodicity we see that the graph of $\eta(t \to 0, \alpha)$ is sawtooth with jumps whenever an energy level passes through zero.

We can apply the spectral asymmetry idea to our continuous Dirac spectrum. Accepting that $2\pi N_\pm^{-1}(k) dn_\pm/dk = 1$, and that we can interchange the order of the integrals over k and x in $Q = \int J^0(x)\,dx$ we anticipate that the scattering states' contribution to $\langle Q \rangle$ can be written as

$$\begin{aligned}\langle Q_{\text{continuum}} \rangle &= -\frac{1}{2}\int_{-\infty}^{\infty} dk \left(\int_{-\infty}^{\infty} dx \left\{ \frac{dn_+}{dk} N_+^{-1}(k)|u_k(x)|^2 - \frac{dn_-}{dk} N_-^{-1}(k)|v_k(x)|^2) \right\} \right) \\ &= -\frac{1}{2}\int_{-\infty}^{\infty} dk \left\{ \frac{dn_+}{dk} - \frac{dn_-}{dk} \right\}.\end{aligned}$$

In other words we claim that

$$\langle Q_{\text{continuum}} \rangle = -\frac{1}{2}\eta(\mathcal{H})$$

where the regulator is unnecessary as the integrals will turn out to be absolutely convergent.

There is again a potential problem with this claim. If we set

$$\mathcal{C} \stackrel{\text{def}}{=} \sigma_3 \mathcal{H} - m\mathbb{I} = \begin{bmatrix} 0 & -i(\partial + V) \\ +i(\partial - V) & 0 \end{bmatrix}$$

we find that

$$\mathcal{C}\mathcal{H} = -\mathcal{H}\mathcal{C}.$$

From this identity it appears that every eigenstate with energy E is transformed by an application of $\mathcal{C}$ into a state of energy $-E$, the one exception

being the bound state ψ_0 which obeys $\mathcal{C}\psi_0 = 0$. The continuous spectrum appears to be symmetric about $E = 0$, making $\eta(\mathcal{H})$ zero. The apparent symmetry is illusory however: the positive and negative continua are *not* symmetric with respect to zero unless the $E = -m$ bound state lies exactly at $E = 0$, *i.e.* in the center of the energy gap. The reason why the symmetry does not hold is that for the notion of "sum over eigenstates" to make sense we need to have well-defined eigenvalue problem. For this we need to impose some suitable self-adjoint boundary conditions at $\pm L$, and having done so $\mathcal{C}$ will generally take a state that satisfies the boundary conditions to one that does not.

To see this failure in action we need to determine what are "suitable" boundary conditions. If

$$\psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix}$$

then the most general self-adjoint boundary conditions for the operator $\mathcal{H}$ are

$$\left.\frac{\psi_1}{\psi_2}\right|_{x=\pm L} = i \tan \Phi_\pm$$

for some angles $\Phi_\pm$. We need to be selective in our choice of these angles in order to ensure that there are no bound states or charge trapped at the boundaries. We can do this by keeping the asymptotic chiral twist angles $\theta(\pm L)$ fixed while for $|x| > L$ scaling the mass scale $\Delta(x) \to \Lambda\Delta(x)$ with Λ very large. This takes $\tanh x \to \Lambda \tanh x$ and $m \to m\Lambda$. Then beyond $x = +L$ all $\epsilon \ll \Lambda m$ solutions decay rapidly as

$$\begin{bmatrix} m \\ i(\kappa+1) \end{bmatrix} e^{-\Lambda\kappa x}, \quad x > L,$$

where $\kappa = \sqrt{m^2+1}$. Similarly, beyond $x = -L$, they decay as

$$\begin{bmatrix} m \\ -i(\kappa+1) \end{bmatrix} e^{+\Lambda\kappa x}, \quad x < -L.$$

The large Λ eigenvalue problem thus becomes the Dirac analogue of a Schrödinger equation for an infinite square-well potential. Requiring the $|x| < L$ solutions to match continuously onto those for $|x| > L$ imposes the self adjoint boundary conditions.

$$\left.\frac{\psi_1}{\psi_2}\right|_{x=L} = \frac{1}{i}\frac{m}{1+\sqrt{1+m^2}},$$

and

$$\left.\frac{\psi_1}{\psi_2}\right|_{x=-L} = -\frac{1}{i}\frac{m}{1+\sqrt{1+m^2}}.$$

A set of positive-energy solutions that satisfy these boundary conditions are the standing waves

$$U_k^{(1)}(x) = \frac{1}{\sqrt{2}}(u_k(x)+u_{-k}(x)) = \frac{1}{\sqrt{\epsilon(m+\epsilon)}}\begin{bmatrix}(m+\epsilon)\cos(kx)\\ i(k\sin(kx)+\cos(kx)\tanh x)\end{bmatrix}, \quad k \geq 0,$$

where the momenta k satisfy

$$\frac{1+k\,\tan(kL)}{(m+\epsilon)} = \frac{1+\sqrt{1+m^2}}{m} \equiv \alpha.$$

We rearrange this condition as

$$Lk = \tan^{-1}\left(\frac{(m+\epsilon)\alpha-1}{k}\right) + n_+^{(1)}\pi.$$

Differentiating (and using Mathematica to simplify the resulting rather nasty expressions) we find that the density of these states is

$$\begin{aligned}\frac{dn_+^{(1)}}{dk} &= \frac{L}{\pi} - \frac{1}{2\pi}\left(\frac{1}{\epsilon^2-m^2} + \frac{\sqrt{1+m^2}}{\epsilon^2} - \frac{m}{\epsilon(\epsilon^2-m^2)}\right)\\ &= \frac{L}{\pi} - \frac{1}{2\pi}\left(\frac{1}{\epsilon(\epsilon+m)} + \frac{\sqrt{1+m^2}}{\epsilon^2}\right), \quad k \geq 0.\end{aligned}$$

There is a second set of standing waves of the form

$$U_k^{(2)}(x) = \frac{1}{\sqrt{2}}(u_k(x)-u_{-k}(x)) = \frac{1}{\sqrt{\epsilon(m+\epsilon)}}\begin{bmatrix}i(m+\epsilon)\sin(kx)\\ k\cos(kx)-\sin(kx)\tanh x\end{bmatrix}, \quad k \geq 0$$

in which k can also be chosen to obey the boundary conditions at $x = \pm L$. The allowed k's are now determined by

$$Lk = \cot^{-1}\left(\frac{1-(m+\epsilon)\alpha}{k}\right) + n_+^{(2)}\pi,$$

leading to

$$\begin{aligned}\frac{dn_+^{(2)}}{dk} &= \frac{L}{\pi} - \frac{1}{2\pi}\left(\frac{1}{\epsilon^2 - m^2} - \frac{\sqrt{1+m^2}}{\epsilon^2} - \frac{m}{\epsilon(\epsilon^2 - m^2)}\right) \\ &= \frac{L}{\pi} - \frac{1}{2\pi}\left(\frac{1}{\epsilon(\epsilon+m)} - \frac{\sqrt{1+m^2}}{\epsilon^2}\right), \quad k \geq 0.\end{aligned}$$

The total density of states for the positive energy continuum is the sum of these[3]

$$\frac{dn_+}{dk} = \frac{1}{\pi}\left\{2L - \frac{1}{\epsilon(\epsilon+m)}\right\}, \quad k \geq 0.$$

Similarly the negative energy states have

$$\frac{dn_-}{dk} = \frac{1}{\pi}\left\{2L - \frac{1}{\epsilon(\epsilon-m)}\right\} \quad k \geq 0.$$

At first sight there is discrepancy of a factor of two between this an our earlier claim that

$$\frac{dn_\pm}{dk} = \frac{1}{2\pi}N_\pm(k),$$

but a complete set of standing waves requires only $k \geq 0$, while the earlier travelling wave solutions had $-\infty < k < \infty$. The present "density of states" has combined both signs of k into one expression.

The $O(L)$ terms cancel when we take difference between the positive and negative density of states. The spectral asymmetry arises solely from the $O(1)$ terms, and is given by the convergent integral

$$\eta(\mathcal{H}) = \frac{1}{\pi}\int_0^\infty dk\left\{-\frac{1}{\epsilon(\epsilon-m)} + \frac{1}{\epsilon(\epsilon+m)}\right\} = \int_{-\infty}^\infty \frac{dk}{2\pi}\left\{\frac{1}{\epsilon(\epsilon+m)} - \frac{1}{\epsilon(\epsilon-m)}\right\}.$$

This integral is not zero. So — as claimed earlier — the symmetry between the positive and negative energy levels suggested by the existence of $\mathcal{C}$ is illusory. Further, examining the integral shows that the induced charge is indeed given by

$$\langle Q_{\text{continuum}}\rangle = -\frac{1}{2}\eta(\mathcal{H}),$$

[3]In keeping with the density of states being the reciprocal of the normalization factors, the $\pm\sqrt{1+m^2}/\epsilon^2$ in the individual density of states correspond to the contribution of the terms involving products of u_k with $u_{-k}^\dagger$ that arise when computing $N(k)$ for $|u_k \pm u_{-k}|^2/2$.

as we anticipated.

Note that the integral of the *sum* of the mass-term-influenced parts of the continuum density of states is

$$\Delta n_{\text{total}} = -\int_{-\infty}^{\infty} \frac{dk}{2\pi} \left\{ \frac{1}{\epsilon(\epsilon+m)} + \frac{1}{\epsilon(\epsilon-m)} \right\} = -1,$$

which shows that exactly one state has been abstracted from the continuous spectrum to become the $E = -m$ in-gap bound state.

Via Green functions: Now consider a different approach, see for example Claudio Chamon *et al.*[4]

Start with our original Hamiltonian

$$\mathcal{H} = \begin{bmatrix} m & \mathcal{D} \\ \mathcal{D}^\dagger & -m \end{bmatrix}$$

where $\mathcal{D} = -i\{\partial_x + V(x)\}$, and $\mathcal{D}^\dagger = -i\{\partial_x - V(x)\}$. Assume that $V(x)$ changes sign as it does in the $V(x) = \tanh(x)$ case.

Let $\chi_\lambda(x)$ be the complete set of orthonormal eigenfunctions of $\mathcal{D}\mathcal{D}^\dagger$ with eigenvalues λ^2 that are obtained by imposing self-adjoint boundary condition (for example that $\chi_\lambda = 0$) at some large distance $x = \pm L$. Then, setting $\epsilon = \sqrt{\lambda^2 + m^2}$, we find that

$$u_\lambda(x) = \frac{1}{\sqrt{2\epsilon(m+\epsilon)}} \begin{bmatrix} (m+\epsilon)\chi_\lambda \\ \mathcal{D}^\dagger \chi_\lambda \end{bmatrix}$$

is an eigenfunction of $\mathcal{H}$ with eigenvalue $+\epsilon$ and

$$v_\lambda(x) = \frac{1}{\sqrt{2\epsilon(\epsilon-m)}} \begin{bmatrix} (m-\epsilon)\chi_\lambda \\ \mathcal{D}^\dagger \chi_\lambda \end{bmatrix}$$

is an eigenfunction of $\mathcal{H}$ with eigenvalue $-\epsilon$.

These 2-spinor Dirac eigenfunctions are also orthonormal. For example, using the hermiticity of $\mathcal{D}^\dagger$, we have

$$\begin{aligned} \int u_\lambda^\dagger(x) u_\mu(x) dx &= \frac{(m+\epsilon)^2}{2\epsilon(m+\epsilon)} \int \chi_\lambda^*(x)\chi_\mu(x)dx + \frac{1}{2\epsilon(m+\epsilon)} \int (\mathcal{D}^\dagger \chi_\lambda(x))^* \mathcal{D}^\dagger \chi_\mu(x) dx \\ &= \delta_{\lambda\mu} \frac{(m+\epsilon)^2 + \lambda^2}{2\epsilon(m+\epsilon)} \\ &= \delta_{\lambda\mu}. \end{aligned}$$

[4] C. Chamon, C-Y. Hou, R. Jackiw, C. Mudry, S-Y. Pi, G. Semenoff, *Electron fractionalization for two-dimensional Dirac fermions*, Phys. Rev. B 77, 235431(2008).

Except for a possible eigenfunction with energy $E = m$ or $E = -m$ that will occur when $\mathcal{D}\mathcal{D}^\dagger$ has a zero mode, these eigenstates come in opposite-energy pairs. In this formulation, therefore, the spectral asymmetry of the continuous spectrum is zero so we should not expect the total system to have a fractional charge[5].

Chamon *et al.* combine the $\pm\epsilon$ eigenvalues to get

$$\rho(x) = -\frac{1}{2}\sum_{\lambda\neq 0}\left(|u_\lambda(x)|^2 - |v_\lambda(x)|^2\right) \;=\; -\frac{1}{2}\sum_{\lambda\neq 0}\frac{m}{\epsilon}\left\{|\chi_\lambda(x)|^2 - \frac{1}{\lambda^2}|\mathcal{D}^\dagger\chi_\lambda(x)|^2\right\}.$$

They then observe that

$$\frac{m}{\epsilon}\left\{|\chi_\lambda(x)|^2 - \frac{1}{\lambda^2}|\mathcal{D}^\dagger\chi_\lambda|^2\right\} = \partial_x\left\{\frac{im}{\epsilon\lambda^2}\chi_\lambda^*(x)\mathcal{D}^\dagger\chi_\lambda(x)\right\}$$

is a total divergence. If we were to integrate $\rho(x)$ over the interval $[-L, L]$, we get zero because the $\chi_\lambda(\pm L) = 0$.. What we can do, however, is to integrate $\rho(x)$ over an interval (a, b) that includes where $V(x)$ is varying, and and with both a and b in regions where $V(x)$ is constant yet still far away from the boundaries.

We need a convenient way to evaluate the integrated-out boundary contribution. To do this we note that

$$\frac{m}{2\epsilon} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{m}{\omega^2 + \lambda^2 + m^2}$$

and use this identity to express the boundary contribution in terms of Green functions

$$\begin{aligned}\left\{\frac{im}{\epsilon\lambda^2}\chi_\lambda^*(x)\mathcal{D}^\dagger\chi_\lambda(x)\right\} &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\langle x|\frac{P}{\mathcal{D}\mathcal{D}^\dagger}\frac{2m(\partial_x - V)}{\mathcal{D}\mathcal{D}^\dagger + \omega^2 + m^2}|x\rangle \\ &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}dx'\langle x|\frac{P}{\mathcal{D}\mathcal{D}^\dagger}|x'\rangle\langle x'|\frac{2m(\partial_x - V)}{\mathcal{D}\mathcal{D}^\dagger + \omega^2 + m^2}|x\rangle.\end{aligned}$$

Here P is a projector away from any $\lambda = 0$ localized zero mode of $\mathcal{D}\mathcal{D}^\dagger$. The derivative in the $\mathcal{D}^\dagger$ in the numerator will not have and effect because it is

[5]These eigenfunctions may not be complete. If the zero mode occurs in $\mathcal{D}^\dagger\mathcal{D}$ instead of in $\mathcal{D}\mathcal{D}^\dagger$ we will have to use the eigenmodes of $\mathcal{D}^\dagger\mathcal{D}$ to get all the stes. This will not affect the fractional charge so I will ignore this complication.

the same at both ends of the interval. As we only want the Green function at large distance where the zero mode has decayed to zero, we can ignore the "P".

A key property of Green functions is that they are *local*: they are not affected by distant boundary conditions or far off potential variations. At large distance

$$\mathcal{D}\mathcal{D}^\dagger \to -\partial_x^2 + |V|^2$$

and here the Green functions become

$$\langle x'|\frac{1}{\mathcal{D}\mathcal{D}^\dagger + m^2 + \omega^2}|x\rangle = \frac{1}{2\sqrt{m^2+V^2+\omega^2}}\exp\{-\sqrt{m^2+V^2+\omega^2}|x-x'|\}$$

$$\langle x|\frac{1}{\mathcal{D}\mathcal{D}^\dagger}|x'\rangle = \frac{1}{2|V|}\exp\{-|V||x-x'|\}.$$

We can evaluate the integrals using our earlier expressions to get

$$\begin{aligned}
& \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}dx'\langle x|\frac{P}{\mathcal{D}\mathcal{D}^\dagger}|x'\rangle\langle x'|\frac{2imV}{\mathcal{D}\mathcal{D}^\dagger+\omega^2+m^2}|x\rangle \\
= \; & \frac{1}{|V|}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{1}{\sqrt{m^2+V^2+\omega^2}}\frac{2mV}{\sqrt{m^2+V^2+\omega^2}+|V|} \\
= \; & \mathrm{sgn}(V)\left(\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left(\frac{|V|}{m}\right)\right),
\end{aligned}$$

and so compute the charge located with the interval (a, b). As we know that the total charge is zero, there must be an equal and oppsite charge located near the boundaries at $x = \pm L$.