Notes on Equivariant Cohomology

One reason for a physicist to be interested in equivariant cohomology is that it provides a language in which to describe and understand the geometric and topological origin of anomalies in gauged non-linear sigma models. The classic paper on this subject is that of Hull and Spence\(^1\) (H&S). Other applications are the equivariant localization techniques for the evaluation of path integrals in terms of the fixed points of a group action. Here a broad review is provided by Szabo\(^2\). An account for mathematicians is the book by Guillemin and Sternberg\(^3\) (G&S). My aim here is to translate the rather abstract account in G&S into language more familiar to physicists and at the same time supply some details that are glossed over in H&S and Szabo.

**Graded Jacobi identities:** As this subject makes considerable use of ideas from supersymmetry, we begin with some generalities about graded algebras. Introduce two families of objects \(B\) (for bosonic) and \(F\) (for fermionic). The bracket is to be the anti-commutator when both elements are in \(F\), the commutator otherwise, and to obey

\[
[B, B] \subseteq B, \\
[B, F] \subseteq F, \\
\{F, F\} \subseteq B.
\]

These inclusions imply that there is a \(Z_2\) grading in which \(|B| = 0\text{ (mod 2)}\) for \(B \in B\), and \(|F| = 1\text{ (mod 2)}\) for \(F \in F\). Using \(\ldots\), we have a graded Jacobi identity

\[
(-1)^{|x||y|}(x, (y, z)) + (-1)^{|x||y|}(y, (z, x)) + (-1)^{|y||z|}(z, (x, y)) = 0,
\]

where \((x, y)\) is the commutator or anti-commutator as appropriate. Explicitly

\[
0 = [B_1, [B_2, B_3]] + [B_2, [B_3, B_1]] + [B_3, [B_1, B_2]], \\
0 = [B_1, [B_2, F_3]] + [B_2, [F_3, B_1]] + [F_3, [B_1, B_2]], \\
0 = [B_1, \{F_2, F_3\}] + \{F_2, [F_3, B_1]\} - \{F_3, [B_1, F_2]\}, \\
0 = [F_1, \{F_2, F_3\}] + [F_2, \{F_3, F_1\}] + [F_3, \{F_1, F_2\}].
\]

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\(^3\)V. W. Guillemin, S. Sternberg *Supersymmetry and Equivariant De Rham theory*. Springer 1999.
No operators are commuted past one another in any of these identities. Consequently, while seeing the pattern is helped by the grading, they hold for all operators.

Using the graded bracket we find that the operation $D_x$ defined by

$$D_x y \overset{\text{def}}{=} (x, y)$$

acts as a graded derivation — meaning that $D_x$ is a derivation or an anti-derivation depending on whether $|x|$ is even or odd. Explicitly

$$[B, O_1 O_2] = [B, O_1]O_2 + O_1[B, O_2],
[F, B_1 B_2] = [F, B_1]B_2 + B_1[F, B_2],
\{F_1, BF_2\} = [F_1, B]F_2 + B\{F_1, F_2\},
\{F_1, F_2 B\} = \{F_1, F_2\}B - F_2[F_1, B],
[F_1, F_2 F_3] = \{F_1, F_2\}F_3 - F_2\{F_1, F_3\}.$$  

Again these formulæ hold for any operators, but the labels $B$ and $F$ help understand the pattern.

**Lie-Cartan superalgebra:** Let $X, Y$ be vector fields on a manifold $M$. Then the contraction $i_X$, the Lie derivative $\mathcal{L}_X$, and the exterior derivative $d$ act as graded derivations on the space $\Omega(M)$ of differential forms. They obey

$$i_X i_Y + i_Y i_X = 0,$$
$$\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]},$$
$$\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]},$$
$$d i_X + i_X d = \mathcal{L}_X,$$
$$d \mathcal{L}_X - \mathcal{L}_X d = 0,$$
$$d^2 = 0.$$

When a set of vector fields form a basis for a Lie algebra $\mathfrak{g} = \text{Lie}(G)$ with Lie brackets

$$[\xi_a, \xi_b] = f_{ab}^c \xi_c,$$
the above relations become

\[ \begin{align*}
  i_a i_b + i_b i_a &= 0, \\
  \mathcal{L}_a i_b - i_b \mathcal{L}_a &= f_{ab}^c i_c, \\
  \mathcal{L}_a \mathcal{L}_b - \mathcal{L}_b \mathcal{L}_a &= f_{ab}^c \mathcal{L}_c, \\
  d i_a + i_a d &= \mathcal{L}_a, \\
  d \mathcal{L}_a - \mathcal{L}_a d &= 0, \\
  d^2 &= 0.
  \end{align*} \]

and extend the Lie algebra to a \emph{Lie superalgebra} \( G^* \). This extended algebra acts on forms and makes \( \Omega(M) \) into a “\( G^* \) module.”

**Weil Model:** We now construct the Weil model for equivariant cohomology as is done in G&S. It is a more abstract example of a \( G^* \) module.

Let the Lie algebra \( g \) over a field have basis \( \xi_a \) so that

\[ [\xi_a, \xi_b] = f_{ab}^c \xi_c, \]

and introduce two sets of “dual basis” objects. A “Bose” set \( \phi^a \) that commute with everything, and a “Fermion” set \( \theta^a \) that mutually anticommute. We introduce a formal antiderivation “\( d \)” operator that acts as

\[ d \theta^a = \phi^a, \quad d \phi^a = 0. \]

For any constant \( c \) in the field we also have \( dc = 0 \). It is clear that this purely algebraic operation obeys \( d^2 = 0 \).

Polynomials in \( \phi^a \) and \( \theta^a \) constitute the algebra \( W \) \( \overset{\text{def}}{=} S(g^*) \otimes \wedge (g^*) \). We introduce formal derivations \( L_a \) that act on the algebra generators as

\[ L_a \phi^b = -f_{ac}^b \phi^c, \quad L_a \theta^b = -f_{ac}^b \theta^c \]

We can verify from the Jacobi identity for \( f_{ac}^b \) that these formal Lie derivatives obey

\[ [L_a, L_b] = f_{ab}^c L_c. \]

We next define an antiderivation contraction operator as

\[ i_a \theta^b = \delta_a^b \]
and desire that \( L_a = i_a d + di_a \). As \( d\delta^a_b = 0 \) this requires

\[
 i_a\phi^b = (i_a d + di_a)\theta^b = L_a\theta^b = -f^b_{ac}\theta^c.
\]

As we have found how \( d, i_a, \) and \( L_a \) act on \( W \), we possess all the ingredients that make \( W \) into a \( G^* \) module.

Now observe that

\[
 \mu^a \overset{\text{def}}{=} d\theta^a + \frac{1}{2} f^a_{jk} \theta^j \theta^k = \phi^a + \frac{1}{2} f^a_{jk} \theta^j \theta^k
\]

obeys

\[
 i_a\mu^b = i_a\phi^b + \frac{1}{2} f^b_{jk} \delta^a_j \theta^k - \frac{1}{2} f^b_{jk} \theta^j \delta^a_k = i_a\phi^b + f^b_{ak} \theta^k
\]

\[
 = 0.
\]

We say that \( \mu^a \) are horizontal. Since the \( \mu^a \) commute, we can take them as the generators of \( S(\mathfrak{g}^*) \) in lieu of the \( \phi^a \). We may verify that

\[
 L_a\mu^b = -f^b_{ac}\mu^c
\]

and

\[
 d\mu^a = -f^a_{bc}\theta^b\mu^c.
\]

The formal \( W \) algebra generators have properties reminiscent of those of gauge fields \( A \) where the curvature two-form \( F = dA + A^2 \) obeys the Bianchi identity \( dF + [A, F] = 0 \). When we write \( A = A^a \lambda_a \) and \( F = F^a \lambda_a \) with matrices \( \lambda_a \) obeying \([\lambda_a, \lambda_b] = f_{ab}^c \lambda_c\), these equations become

\[
 F^a = dA^a + \frac{1}{2} f^a_{bc} A^b A^c
\]

\[
 0 = dF^a + f^a_{bc} A^b F^c,
\]

and suggest the identification

\[
 \theta^a \leftrightarrow A^a,
\]

\[
 \mu^a \leftrightarrow F^a.
\]

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In later sections will explain in what sense this identification holds.

We now extend Weil model to the space $W \otimes \Omega(M)$ with

$$d \rightarrow d \otimes \text{id} + \text{id} \otimes d.$$ 

Here “id” is the identity operation on its factor in the tensor product, and the “$d$” in $\text{id} \otimes d$ is the usual exterior derivative on $p$-forms. We also extend

$$i_a \rightarrow i_a = i_a \otimes \text{id} + \text{id} \otimes i_a$$

$$L_a \rightarrow L_a = L_a \otimes \text{id} + \text{id} \otimes L_a,$$

where the $i_a$ in $\text{id} \otimes i_a$ is the insertion of the vector field $\xi$ into a form in $\Omega(M)$. Similarly $L_a \equiv L_{\xi_a}$ is the usual Lie derivative on forms.

The equivariant cohomology of $M$ is now defined as the cohomology of $d$ restricted to the subspace of to basic forms. These are both horizontal, meaning that $\iota_a \omega = 0$, and equivariant i.e. $L_a \omega = 0$.

**Cartan model:** The Cartan model processes the same information as the Weil model, but implements the familiar Riemann-geometry trick of reducing labour by utilizing normal co-ordinates in which the Christoffel symbols vanish at the point of interest. The equivalent technique in a gauge theory is to select a gauge in which $A = 0$ at that point. We can then set $A \rightarrow 0$, provided we do so only after taking all derivatives.

In this spirit Cartan sets $\theta^a \rightarrow 0$ so that that our $G^*$ module reduces to $S(g^*) \otimes \Omega(M)$ whose elements are sums of objects such as

$$\omega = \mu_1 \cdots \mu_p \omega_{a_1 \cdots a_p} = \frac{1}{q!} \mu^{a_1} \cdots \mu^{a_p} \omega_{a_1 \cdots a_p, \mu_1 \cdots \mu_q} (x) dx^{\mu_1} \cdots dx^{\mu_q},$$

with grading $2p + q$. Take as Lie derivative

$$L_a \equiv L_a \otimes \text{id} + \text{id} \otimes L_a$$

where, as before,

$$L_a \mu^b = - f^{bc}_{ac} \mu^c,$$

but now $d \mu^a = 0$ as $\theta^a$ has been set to zero.

Again equivariant forms are those such that $L_a \omega = 0$. This means that the ordinary base-space Lie derivative of a Cartan-model equivariant form must obey

$$L_a \omega_{a_1 \cdots a_p} = f^c_{a a_1} \omega_{ca_2 \cdots a_p} + f^c_{a a_2} \omega_{a_1 ca_3 \cdots a_p} + \cdots$$
The Cartan-model exterior derivative is now defined to be

\[ d_C = \text{id} \otimes d - \mu^a \otimes i_a. \]

This definition is motivated from the Weil derivative by noting that \( d\theta^a \sim \mu^a \) once the \( \theta^a \) are set to zero after differentiating. Consequently \( d \otimes \text{id} \sim \mu^a i_a \otimes \text{id} \). Now when acting on on Weil-model horizontal forms we have \( i_a \otimes \text{id} = -\text{id} \otimes i_a \), so on such forms we have

\[ \text{id} \otimes d + d \otimes \text{id} \rightarrow \text{id} \otimes d - \mu^a \otimes i_a. \]

Observe that

\[ d_C^2 = -\mu^a \otimes (i_a d + di_a) = -\mu^a \otimes \mathcal{L}_a. \]

Now \( \mathcal{L}_a \) acts on \( L_a \) equivariant forms \( \omega = \mu^a_1 \cdots \mu^a_p \omega_{a_1 \ldots a_p} \) to give

\[ \mu^a \otimes \mathcal{L}_a \omega = \mu^a \mu^{a_1} \cdots \mu^{a_p} (f^c_{a_1} \omega_{a_2 \ldots a_p} + f^c_{a_2} \omega_{a_1 a_3 \ldots a_p} + \cdots) = 0. \]

The vanishing occurs because of the antisymmetry of the \( f^a_{bc} \) coupled with the commuting property of the \( \mu^a \). This point is rather rushed over in Szabo’s review.

We also observe that

\[
\begin{align*}
[L_a, d_C] &= [L_a \otimes \text{id}, -\mu^b \otimes i_b] + [\text{id} \otimes \mathcal{L}_a, -\mu^b \otimes i_b] \\
&= f^b_{ac} \mu^c \otimes i_b - \mu^b \otimes [\mathcal{L}_a, i_b] \\
&= f^c_{ab} \mu^b \otimes i_c - \mu^b f^c_{ab} \otimes i_c \\
&= 0,
\end{align*}
\]

so \( d_C \) preserves the equivariance property and we can use \( d_C \) to define an equivariant cohomology.

Despite the plausible motivation from the Weil algebra, it is not immediately obvious that the Weil model and the Cartan model result in the same cohomology. That it is so, is established via the Mathai-Quillen isomorphism.

**Principal and associated bundles:** The abstract constructions of the previous sections seem unmotivated, but their origin becomes clear when we seek to associate a bundle \( \pi : E \rightarrow X \) whose fibre \( \pi^{-1}(x) \) is a copy of the manifold \( M \) to a principal bundle \( P \). To understand this process it helps if
we first review the Ehresmann construction that associates a vector space acted on by Lie group $G$ to a principal $G$-bundle, and so outputs a bundle with $G$ as its gauge group.

Start with a Lie group $G$ and faithful representation $G : V \to V$. Write the action of $g \in G$ on $v \in V$ as

$$g : v \mapsto g v.$$

After choosing a basis for $V$ we will have representation matrices $g \mapsto D_{ij}(g)$ and the generators of $\mathfrak{g} = \text{Lie}(G)$ will become matrices $(\lambda_a)_{ij}$ obeying

$$[\lambda_a, \lambda_b] = f^c_{ab} \lambda_c.$$

Often I will not distinguish between the abstract generators of $\mathfrak{g}$ and these matrices, nor between $g \in G$ and its corresponding matrix $D(g)$.

The group $G$ acts on itself by multiplication on either the right or the left. As a consequence the Lie algebra $\mathfrak{g}$ of infinitesimal group elements equips the group manifold of $G$ with left- and right-invariant vector fields, $\xi_L^a$ and $\xi_R^a$ respectively that have Lie brackets

$$[\xi_L^a, \xi_L^b] = f^c_{ab} \xi_L^c, \quad [\xi_R^a, \xi_R^b] = -f^c_{ab} \xi_R^c.$$

Recall that $\xi_L^a$ corresponds to multiplication by infinitesimal elements $(1 + \epsilon \lambda_a)$ from the right and $\xi_R^a$ to infinitesimal multiplication from the left.

Now let $\pi : P \to X$ be a principal bundle with fibre $G$. Given a local trivialization $\pi^{-1}(U) = U \times G$ over $U \subseteq X$, we use co-ordinates $(x, g)$ with $x \in U$ and $g \in G$.

A connection is defined by introducing a covariant derivative, which is a vector field on $P$ of the form

$$D_\mu = \left( \frac{\partial}{\partial x^\mu} \right)_g - A^a_\mu(x) \xi_R^a.$$

As right-invariant fields arise from an infinitesimal multiplication on the left, a parameterized flow along this vector field is one in which

$$\frac{dg}{ds} = -A^a_\mu(x) \lambda_a g \frac{dx}{ds}.$$

Solving this equation lifts a curve $x(s)$ in the base space to a curve $(x(s), g(s))$ in the total space $P$. 

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The Lie bracket of these vector fields is

$$[D_\mu, D_\nu] = -\mathcal{F}^a_{\mu\nu} \xi^R_a,$$

where

$$\mathcal{F}^a_{\mu\nu}(x) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu.$$

Figure 1: The decomposition of the tangent space $T_p(P) = H_p(P) \oplus V_p(P)$. The one-form called $\omega_p$ in this figure from Wikipedia is the same as our $\mathcal{A}$ evaluated at $p \in P$.

At $p = (x, g) \in P$, the $D_\mu$ span the horizontal subspace $H_p(P) \subset T_p(P)$ while the $\xi^L_a$ (or the $\xi^R_a$) span the vertical subspace $V_p(P)$ — i.e. the directions along the fibre. Thus we have a decomposition

$$T_p(P) = H_p(P) \oplus V_p(P).$$

Because the right-invariant vector fields correspond to infinitesimal left actions, this decomposition is invariant under a global (meaning that $h$ is independent of $x$) push forward $h : (x, g) \mapsto (x, gh)$.

The matrix-valued left- and right-invariant Maurer-Cartan forms on each fibre are

$$\omega_L = \omega_L^a \lambda_a = g^{-1} \delta g, \quad \omega_R = \omega_R^a \lambda_a = \delta gg^{-1},$$

where $\delta$ is the exterior derivative on $G$ and “$g$” is shorthand for the matrix $D_{ij}(g)$ that represents $g \in G$ in the representation $V$. 
We define the matrix-valued forms
\[ A = \lambda_a \omega^a_{\mu} dx^\mu, \quad F = \frac{1}{2} \lambda_a \omega^a_{\mu\nu} dx^\mu dx^\nu, \]
and from them
\[ A(x, g) = g^{-1} \delta g + g^{-1} A(x) g, \quad F(x, g) = g^{-1} F(x) g. \]

Notwithstanding its definition in our local trivialization, the connection form \( A \) is defined \textit{globally} on \( P \) in a coordinate independent fashion by its evaluation on the vector fields
\[ A(\xi^L_a) = \lambda_a, \quad A(\mathcal{D}_\mu) = g^{-1}(-A^a_\mu \lambda_a + A^a_\mu \lambda_a)g = 0. \]
The term from \( g^{-1} dg \) in the second line arises because the flow along \( \mathcal{D}_\mu \) has displacements \( dg = -A^a_\mu \lambda_a dx \).

If \( d = d_P \equiv d_X + \delta \) with \( d_X \delta + \delta d_X = 0 \), we can use the Maurer-Cartan relation \( \delta(g^{-1} \delta g) = -(g^{-1} \delta g)^2 \) to show that
\[ \mathbb{F} = dA + A^2, \quad 0 = d\mathbb{F} + [A, \mathbb{F}]. \]

Let \( L_a \) be the Lie derivative on \( G \) (and hence on \( P \)) with respect to the \textit{left} invariant field \( \xi^L_a \), then from \( \omega^a_L(\xi^L_b) = \delta^a_b \) and the derivation property of the Lie derivative
\[ (L_a \omega^b_L)(\xi^L_b) + \omega^b_L([\xi^L_a, \xi^L_b]) = L_a(\omega^b_L(\xi^L_b)) = 0, \]
we have
\[ L_a \omega^b_L = -f^b_{ac} \omega^c_L. \]
We also find
\[ L_a g = g \lambda_a, \quad L_a g^{-1} = -\lambda_a g^{-1}, \]
and so
\[ L_a A^b = -f^b_{ac} A^c, \quad L_a F^b = -f^b_{ac} F^c. \]
With these last two equations in hand, we see that the globally defined \( A \) and \( F \) possess all of the properties required of the abstract quantities \( \theta^b \) and \( \mu^b \) of the Weil algebra.
The associated vector bundle has total space space \( E = P \times_G V \) which is defined to be the quotient of \( P \times V \) under the identification

\[
(x, gh, v) \sim (x, g, hv),
\]
or equivalently

\[
(x, g, v) \sim (x, gh, h^{-1}v).
\]
Sections of the associated bundle are choices of \( v(x, g) \in V \) for each \( p = (x, g) \in P \) that are compatible with this identification. As, for fixed \( h \), the set \( (x, g, v(x, g)) \) coincides with \( (x, gh, v(x, gh)) \) we must have

\[
(x, gh, h^{-1}v(x, g)) = (x, gh, v(x, gh))
\]
so the compatibility condition is

\[ v(x, gh) = h^{-1}v(x, g). \]

When the complex-valued fields \( \psi_i(x, g) \) are the components of a section \( v(x, g) \) this condition becomes

\[ \psi_i(x, gh) = D_{ij}(h^{-1})\psi_j(x, g), \]
where the \( D_{ij}(g) \) are the matrices representing \( G \). We will regard any set of functions \( \psi_j(x, g) \) possessing this transformation property as being a section of the bundle.

We can now make contact with a more familiar definition of a covariant derivative. We begin by recalling that right invariant vector fields are derivatives that involve infinitesimal multiplication from the left. Their definition is

\[
\xi^R_a \psi_i(x, g) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (\psi_i(x, (1 + \epsilon \lambda_a)g) - \psi_i(x, g)),
\]
where \([\lambda_a, \lambda_b] = f_{abc}^b \lambda_c \).

As \( \psi^j(x, g) \) is a section of the associated bundle, we know how it varies when we multiply group elements in on the right. We therefore write

\[
(1 + \epsilon \lambda_a)g = g g^{-1}(1 + \epsilon \lambda_a)g,
\]
and from this, (and, as usual, writing \( g \) for \( D(g) \) where it makes for compact notation) we find

\[
\xi^R_a \psi_i(x, g) = \lim_{\epsilon \to 0} (D_{ij}(g^{-1}(1 - \epsilon \lambda_a)g)\psi_j(x, g) - \psi_i(x, g)) / \epsilon
\]

\[
= -D_{ij}(g^{-1})(\lambda_a)_{jk}D_{kl}(g)\psi_l(x, g)
\]

\[
= -(g^{-1}\lambda_a g)_{ij}\psi_j(x, g).
\]
Acting on sections, we therefore have
\[ D_\mu \psi = (\partial_\mu \psi)_g + (g^{-1} A_\mu g) \psi. \]

This still does not look too familiar, because the derivatives with respect to \( x_\mu \) are being taken at fixed \( g \). We normally fix a gauge by making a choice of \( g = \sigma(x) \) for each \( x_\mu \). The conventional wavefunction \( \psi(x) \) is then \( \psi(x, \sigma(x)) \).

We can use \( \psi(x, \sigma(x)) = \sigma^{-1}(x) \psi(x, e) \), to obtain
\[ \partial_\mu \psi = (\partial_\mu \psi)_\sigma + (\sigma^{-1}) \sigma \psi = (\partial_\mu \psi)_\sigma - (\sigma^{-1} \partial_\mu \sigma) \psi. \]

From this, we get a derivative
\[ \nabla_\mu \overset{\text{def}}{=} \partial_\mu + (\sigma^{-1} A_\mu \sigma + \sigma^{-1} \partial_\mu \sigma) = \partial_\mu + A_\mu, \]
on functions \( \psi(x) \overset{\text{def}}{=} \psi(x, \sigma(x)) \) defined (locally) on the base space \( X \). This is the conventional physicists’ covariant derivative, now containing gauge fields
\[ A_\mu(x) = \sigma^{-1} A_\mu \sigma + \sigma^{-1} \partial_\mu \sigma \]
that are gauge transformations of our \( g \)-independent \( A_\mu \). The derivative has been constructed so that
\[ \nabla_\mu \varphi(x) = D_\mu \varphi(x, g) \big|_{g=\sigma(x)}, \]
and has commutator
\[ [\nabla_\mu, \nabla_\nu] = \sigma^{-1} F_{\mu\nu} \sigma = F_{\mu\nu}. \]
Note the sign change vis-à-vis the commutator of the \( D_\mu \) on the total space.

It is the curvature tensor \( F_{\mu\nu} \) that is usually met with in physics. Recall that it provides a matrix- or Lie-algebra-valued two-form
\[ F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = dA + A^2 \]
on the base space where the connection \( A = A^a \lambda_a dx^\mu = A_\mu dx^\mu \) is a Lie-algebra-valued one-form on the base space. Both \( F \) and \( A \) have been defined only in the open region \( U \subset M \) in which the smooth gauge-choice section \( \sigma(x) \) has been selected. They will need to be patched together on overlaps.
**Associated $M$-bundles:** We now wish to examine the analogous objects on the bundle $\pi : P \times_G M \to X$, which we regard as an associated bundle with its fibre being the manifold $M$. We will write the points on $M$ as $\varphi$ and their co-ordinates as $\varphi^i$. Points in the total space of the new bundle are equivalence classes of triples $(x, g, \varphi) \sim (x, gh, h^{-1}\varphi)$. The primary difference between this case and the vector bundle case is that the infinitesimal left action on $M$ is not given by a set of constant matrices but by position-dependent vector fields. Essentially we are gauging a non-linear sigma model, and so we follow the basic strategy of (H&S). I will, however, use the definitions in Guillemin and Sternberg which differ in significant details.

We begin by constructing the analogue of the Lie algebra representation matrices $(\lambda_a)_{ij}$. We consider a left action of $G$ on $M$ in which $g : \varphi \mapsto g\varphi$, $g_2(g_1\varphi) = (g_2g_1)\varphi$.

If the near-identity group element $g = 1 + \epsilon^a\lambda_a$ maps

$$\varphi^i \rightarrow \varphi^i + \epsilon^a X_a^i(\varphi)$$

Then the product $(1 + \epsilon^b\lambda_b)(1 + \epsilon^a\lambda_a)$, in which the rightmost operation is performed before the left one, takes

$$\varphi^i \rightarrow \varphi^i + \epsilon^a X_a^i(\varphi) + \epsilon^b X_b^i(\varphi) + \epsilon^a \epsilon^b X_a^i \partial_j X_b^j.$$

(Here I have ignored some unimportant terms of $O[\epsilon^2]$ while keeping the only significant one.) Now we perform the same operations in the opposite order. Remembering that

$$[X_a, X_b]^i = X_a^j \partial_j X_b^i - X_b^j \partial_j X_a^i$$

shows that the Lie algebra commutator $[\lambda_b, \lambda_a] \in g$ corresponds to the Lie bracket $X_{[\lambda_b, \lambda_a]} = [X_a, X_b] = -[X_b, X_a]$. This annoying minus sign is the source of an irritancy$^4$ of some formulæ in H&S.

The group action on $M$ induces a representation $\rho(g)$ on $C^\infty[M]$ functions by

$$g : F[\varphi] \mapsto \rho(g)F[\varphi] = F[g^{-1}\varphi].$$

$^4$Invalidation. E.g “The tinsel of the feu is an irritancy incident to every feu-right, by the failure to pay the feu-duty”, William Bell, Dictionary and Digest of the Law of Scotland: With Short Explanations of the Most Ordinary English Law Terms (1838) p995.
We will usually drop the “$\rho$” and just write $gF[\varphi]$. The $g^{-1}$ is necessary so that
\[ g_2(g_1F)[\varphi] = (g_1F)[g_2^{-1}\varphi] = F[g_1^{-1}(g_2^{-1}\varphi)] = F[(g_2g_1)^{-1}\varphi] = (g_2g_1)F[\varphi]. \]
Thus
\[ (1 + \epsilon^a\lambda_a)F = F - \epsilon^aX^i_a\partial_i F = (1 - \epsilon^aX^i_a\partial_i)F. \]
To cope with the minus sign G&S define the fundamental vector fields associated to the generators $\lambda_a \in \mathfrak{g}$ to be
\[ \xi_a = -X^i_a(\varphi)\partial_i, \quad \partial_i \equiv \frac{\partial}{\partial\varphi^i}. \]
Then if $[\lambda_a, \lambda_b] = f^c_{ab}\lambda_c \in \mathfrak{g}$ we use the sign-reversal in the commutator $\leftrightarrow$ Lie-Bracket correspondence to get
\[ [\xi_a, \xi_b] = [-X_a, -X_b] = f^c_{ab}(-X_c) = f^c_{ab}\xi_c. \]
With $\text{Ad}(g)\lambda = g\lambda g^{-1}$ we have
\[ \rho(g)L_{\xi}\rho^{-1}(g) = L_{\text{Ad}(g)\xi} \]
\[ \rho(g)i_{\xi}\rho^{-1}(g) = i_{\text{Ad}(g)\xi} \]
together with the other Lie superalgebra formula given earlier.

Recall that, with local coordinates $(x, g)$ for $P$, the associated bundle $P \times_G M$ has coordinates $(x, g, \varphi)$ and an equivalence relation $(x, gh, \varphi) \sim (x, g, h\varphi)$. Functions on $P \times_G M$ therefore obey
\[ F[x, gh, \varphi] = F[x, g, h\varphi]. \]
On recalling that $F(h\varphi) = \rho(h^{-1})F(\varphi)$, we see that sections of the associated bundle are functions on the total space $P \times M$ such that
\[ F(x, gh, \varphi) = \rho(h^{-1})F(x, g, \varphi) \]
and therefore closely mimic the vector bundle case.

Again a gauge choice is a local section $g = \sigma(x)$ of $P$ and given given a mapping from the base-space $X$ to $M$ we pull back functions $F(x, \sigma(x), \varphi(x)) \equiv F(x, \varphi(x))$ to $X$ and can introduce a covariant derivative by the same process as for a vector bundle:
\[ \nabla_{\mu}F(\varphi(x)) = \partial_{\mu} + A^a_{\mu}(x)\xi_a \]
where

\[ A_\mu(x) = \sigma^{-1}A_\mu\sigma + \sigma^{-1}\partial_\mu\sigma. \]

Then

\[ [\nabla_\mu, \nabla_\nu] = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + A^b_\mu A^c_\nu f_{bc}^a)\xi_a, \]

\[ \equiv F^a_{\mu\nu}\xi_a. \]

H&S define an object with the opposite sign

\[ D_\mu\varphi^i \overset{\text{def}}{=} \partial_\mu\varphi^i - A^a_\mu(x)\xi_a^i, \]

which they also call a “covariant derivative.” There is a problem with this interpretation. If we naïvely interpret \( D_\mu \) example of a derivative acting on functions (in this case \( F \))

\[ D_\mu F(x, \varphi) = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + A^b_\mu A^c_\nu f_{bc}^a)\xi_a \]

then using the Lie bracket \([\xi_a, \xi_b] = f^c_{ab}\xi_c \) we would have

\[ [D_\mu, D_\nu]F = -(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^a_{bc}A^b_\mu A^c_\nu)\xi_a^i \partial_i F. \]

This has the wrong sign before the \( f^a_{bc}A^b_\mu A^c_\nu \) term in the curvature. We must therefore regard \( \varphi^i \) as a “point” on \( M \) rather than a function.

The key property of \( D_\mu \) is that under the replacement \( \varphi^i \to \varphi^i_g = \varphi^i - \epsilon^a\xi_a^i \) in both terms\(^5\), we find that

\[ D_\mu\varphi^i \to (\delta^i_j - \epsilon^a\partial_j\xi_a^i)\left( \partial_\mu \varphi^j - (A^a_\mu + \partial_\mu\epsilon^a + f^b_{cd}A^c_\mu \epsilon^d)\xi_b^j \right) + O(\epsilon^2). \]

**Details:** Working to \( O(\epsilon) \) we have

\[
\partial_\mu(\varphi^i - \epsilon^a\xi_a^i[\varphi]) \to \partial_\mu(\varphi^i - \epsilon^a\xi_a^i) - A^a_{\mu\xi_a^i}[\varphi - \epsilon^b\xi_b]
\]

\[ = (\partial_\mu \varphi^i - A^a_\mu \xi_a^i) - \xi_a^i \partial_\mu \epsilon^a - \epsilon^a\partial_\mu \varphi^i \partial_j \xi_a^i + A^a_\mu (\epsilon^a \xi_b^j \partial_j \xi_a^i)
\]

\[ = (\partial_\mu \varphi^i - A^a_\mu \xi_a^i) - \xi_a^i \partial_\mu \epsilon^a - \epsilon^a\partial_\mu \varphi^i \partial_j \xi_a^i + A^a_\mu \epsilon^b (f^c_{ba} \xi_b + \xi_j \partial_j \xi_a^i)
\]

\[ = (\partial_\mu \varphi^i - A^a_\mu \xi_a^i) - \xi_a^i \partial_\mu \epsilon^a - \epsilon^a\partial_\mu \varphi^i \partial_j \xi_a^i + A^a_\mu \epsilon^b f^c_{ba} \xi_c + A^b_\mu \epsilon^c \xi_b \partial_j \xi_a^i
\]

\[ = (\partial_\mu \varphi^i - A^a_\mu \xi_a^i) - \xi_a^i \partial_\mu \epsilon^a - \epsilon^a\partial_\mu \varphi^i \partial_j \xi_a^i + A^a_\mu \epsilon^b f^c_{ba} \xi_b + A^b_\mu \epsilon^c \xi_b \partial_j \xi_a^i
\]

\[ = (\partial_\mu \varphi^i - A^a_\mu \xi_a^i) - \xi_a^i \partial_\mu \epsilon^a - \epsilon^a\partial_\mu \varphi^i \partial_j \xi_a^i + A^a_\mu \epsilon^b f^c_{ba} \xi_b - A^a_\mu \epsilon^b f^b_{cd} \xi_d + A^b_\mu \epsilon^c \xi_b \partial_j \xi_a^i
\]

\[ = (\delta^i_j - \epsilon^a \partial_j \xi_a^i) \left[ \partial_\mu \varphi^j - (A^a_\mu + \partial_\mu \epsilon^a + f^b_{cd}A^c_\mu \epsilon^d)\xi_b^j \right] + O(\epsilon^2)
\]

\(^5\)Note the sign difference with H&S’s eq (2.10).
We have made use of the Lie bracket

$$\xi^j b \partial_j \xi^i - \xi^j a \partial_j \xi^i = f^c_{ba} \xi^i.$$ 

The transformation

$$A^b_{\mu} \rightarrow A^b_{\mu} + \partial_\mu \epsilon^b + f^b_{cd} A^c \epsilon^d$$

is the infinitesimal form of

$$A \rightarrow A^g = g^{-1} d g + g^{-1} A g, \quad g = (1 + \epsilon^b \lambda_b), \quad [\lambda_a, \lambda_b] = f^c_{ab} \lambda_c,$$

and implies that — despite the minus sign and the $\xi^a$’s having left-invariant commutators — the curvature is still consistent with $F = dA + A^2$.

We can interpret the transformation as

$$\frac{\partial (\phi^i + \delta \phi^i)}{\partial \phi^j} D_{A^g} \phi^j = D_A (\phi^i + \delta \phi^i), \quad \delta \phi^i = -\epsilon^a \xi_a^i.$$ 

For a finite transformation $g(t) = \exp \{ -t \epsilon^a \xi_a \} = \exp \{ +t \epsilon^a X_a \}$ we have

$$D_A (g \phi^i) = \frac{\partial (g \phi)^i}{\partial \phi^j} D_{A^g} \phi^j.$$ 

We can think of this as a generalization of the manner in which a one-form changes under a map $\phi^i \rightarrow f^i (\phi)$

$$d (f^i (\phi)) = \frac{\partial f^i (\phi)}{\partial \phi^j} d \phi^j$$

but made covariant by

$$g^{-1} D_A (g \phi) = D_{A^g} \phi.$$ 

Thus $D \phi^i$ should be thought of a covariant one-form rather than a covariant derivative. In particular the $g$ mapping is

$$\phi^i (0) \rightarrow g(t) \phi^i (0) = \phi^i (t), \quad \frac{d}{dt} \phi^i (t) = -\epsilon^a \xi_a^i [\phi (t)],$$

which is the flow that corresponds to the group action on the points in $M$ rather than on functions.

**Mathai-Quillen in Hull and Spence:** For a $p$-form

$$T = \frac{1}{p!} T_{i_1 \ldots i_p} [\phi] d \phi^{i_1} \cdots d \phi^{i_p} \in \Omega (M),$$
H&S define the “covariant pull-back” from $M$ to space-time $X$ as
\[ \bar{T} \equiv \bar{\varphi}^*(T) \equiv \frac{1}{p!} T_{i_1 \ldots i_p} [\varphi(x)] D_{\mu_1} \varphi^{i_1} \cdots D_{\mu_p} \varphi^{i_p} dx^{\mu_1} \cdots dx^{\mu_p} \in \Omega(X). \]

They then assert that
\[ d(\bar{T}) = \bar{dT} - F^a(\bar{i}_a \bar{T}) + A^a(\mathcal{L}_a \bar{T}). \quad (\star) \]

The identity $(\star)$ is a manifestation of the Mathai-Quillen isomorphism (G&S Theorem 4.1.1) that connects the Weil algebra with the Cartan model.

**Proof of assertion:** On functions $T[\varphi]$ we have $dT = \partial_i T \, d\phi^i$ and
\[ d(\bar{T}) = d(T[\varphi(x)]) = \partial_\mu \varphi^i \partial_i T \, dx^\mu. \]

This coincides with the sum of
\[ \bar{dT} = \partial_i T(\partial_\mu \varphi^i - A^a_\mu \xi_a^i) dx^\mu, \]
\[ -F^a(\bar{i}_a \bar{T}) = 0, \]
\[ A^a_j(\mathcal{L}_a \mathcal{T}) = A^a_\mu \xi_a^j \partial_i T \, dx^\mu, \]

so it works on functions.

On one forms
\[ T = T_i d\phi^i, \quad dT = \partial_i T_j \, d\phi^i \, d\phi^j, \]
\[ \bar{T} = T_i (\partial_\nu \varphi^i - A^a_\nu \xi_a^i) \, dx^\nu, \]
and
\[ \bar{dT} = \partial_j T_j (\partial_\nu \varphi^i - A^a_\nu \xi_a^i) (\partial_\rho \varphi^j - A^b_\rho \xi_b^j) \, dx^\mu \, dx^\nu, \]
\[ -F^a(\bar{i}_a \bar{T}) = -\frac{1}{2} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu) T_i \xi_a^i \, dx^\mu \, dx^\nu, \]
\[ A^a_j(\mathcal{L}_a T) = A^a_\mu (\xi_a^i \partial_i T_j + T_i \partial_j \xi_a^i) (\partial_\nu \varphi^j - A^b_\nu \xi_b^j) \, dx^\mu \, dx^\nu. \]

The sum of these three lines is to be compared with
\[ d(\bar{T}) = \{ \partial_\mu \varphi^i \partial_i T_j (\partial_\nu \varphi^j - A^a_\nu \xi_a^j) - T_j (\partial_\mu A^a_\nu \xi_a^i) \xi_a^j - T_i A^a_\nu \partial_\mu \varphi^j \partial_j \xi_a^i \} \, dx^\mu \, dx^\nu, \]

On using $[\xi_a, \xi_b] = +f^a_{ab} \xi_c$, we find that they agree. As the assertion is true on both functions and one-forms, and as both sides are anti-derivations, it is true for all $p$-forms.
Now suppose that
\[ \omega = \omega^{(2N)} + \mu^{a_1} \omega^{(2N-2)} + \cdots + \mu^{a_1 \cdots a_N} \omega^{(0)} \]
is a form on \( S[g^*] \otimes \Omega(M) \) that is a Cartan-model closed
\[ 0 = d_C \omega = (id \otimes d - \mu^a \otimes i_a) \omega, \]
and Cartan-model equivariant
\[ 0 = L_\omega \omega = (L_a \otimes \text{id} + \text{id} \otimes L_\omega) \omega. \]
Here the numbers in parenthesis indicate the degree of the form on \( M \). We extend the definition of the covariant pull-back to include \( \mu^a \mapsto F^a \) and so obtain
\[ \omega = \omega^{(2N)} + F^{a_1} \omega^{(2N-2)} + \cdots + F^{a_1 \cdots a_N} \omega^{(0)} \in \Omega(X). \]
Now recall that \( L_a \mu^b = -f^b_{ac} \mu^c \) and
\[ dF^b = -f^b_{ac} A^a F^c \]
so
\[ d(\tilde{\omega}) = d\omega - F^a \omega^{(1)} \]
where the additional \( L_a \otimes \text{id} \) that converts \( L_a \) to \( L_{\tilde{\omega}} \) in the last term comes from the \( d \) acting on the \( F^a \)'s in \( d(\tilde{\omega}) \). If \( \omega \) is \( L \) equivariant the last term is zero, and if \( \omega \) is \( d_C \)-closed, the first two terms add to zero. Thus the section \( \tilde{\omega} \) is a conventionally-closed \( p \)-form on \( X \). The extended covariant pull-back has cleverly reinserted the \( A^{\mu} \) that had been set to zero in the Cartan model.

Hull and Spence point out that the task of extending a Wess-Zumino term in an odd number \( (2n + 1) \) of dimensions has a potential “sting” in its tail. Let \( \omega^{(2n+1)} \in Z^{2n+1}(M) \) be a WZ term for some \( 2n \) dimensional theory on \( \partial M \), for which we seek an equivariant extension
\[ \omega^\# = \omega^{(2n+1)} + F^{a_1} \omega^{(2n-1)} + \cdots + F^{a_1 \cdots a_n} \omega^{(1)} \omega_{a_1 \cdots a_n}. \]
Suppose also we have satisfied all Lie derivative conditions, and all cohomological conditions except the last. Then
\[ d\omega^\# = -F^a F^{a_1} \cdots F^{a_n} i_a \omega^{(1)} \]
where we have made explicit the symmetrization over the indices induced by the \( F^a \) commuting. Now one of these earlier conditions is that
\[ d\omega^{(1)}_{a_1 \cdots a_n} = i_{(a_1} \omega^{(3)}_{a_2 \cdots a_n)} \]
and another is
\[
0 = \mathcal{L}^\text{cov}_{a} \omega^{(1)}_{a_1...a_n} = (i_a d + di_a)\omega^{(1)}_{a_1...a_n} - f^b_{a_1} \omega^{(1)}_{b_1...a_n} + \cdots \\
= i_a i_{(a_1} \omega^{(3)}_{a_2...a_n)} + d(i_a \omega^{(1)}_{a_1...a_n}) - f^b_{a_1} \omega^{(1)}_{b_2...a_n} + \cdots
\]
Symmetrizing over all subscripts and using \(i_a i_b + i_b i_a = 0\) shows that
\[
di_{(a} \omega^{(1)}_{a_1...a_n)} = 0.
\]
so the 0-forms \(c_{a_0...a_n} = i_{(a} \omega^{(1)}_{a_1...a_n)}\) are constants, and furthermore by the covariant Lie derivative condition, they are group-invariant tensors. Consequently
\[
d\omega^\# = -c_{a_0...a_n} F^{a_0} F^{a_1} \cdots F^{a_n} = C_{2n+2}
\]
is a characteristic class in two-higher dimensions than the boundary theory. If we write it as \(C_{2n+2} = d(CS_{2n+1})\) we have that
\[
W = \omega^\# + CS_{2n+1}
\]
is closed, but not gauge invariant, and with its anomaly induced from the \(2n + 2\) dimensional characteristic class.

**Equivariant versus Invariant.** We want physical objects to be gauge invariant, so why do mathematicians use the word “equivariant”? Their idea is that a Cartan-model \(p\)-form
\[
\alpha = \mu^{a_1} \cdots \mu^{a_N} \alpha^{(p)}_{a_1...a_N}
\]
defines a “symmetric polynomial function” on the Lie algebra in the sense that if \(X = X^a \lambda_a\) is an element of the Lie algebra then we set \(\mu^{a}(X) = X^a\) and so
\[
\alpha(X) = X^{a_1} \cdots X^{a_N} \alpha^{(0)}_{a_1...a_N}
\]
becomes an ordinary differential form in \(\Omega(M)\) that is confusingly also called \(\alpha\). Thus we have three things with the same name: a Cartan-model form, a mapping to \(\Omega(M)\), and the resulting element of \(\Omega(M)\).

We recall our \(G\) action on forms in \(\Omega(M)\) that is given by \(g : \alpha \mapsto g \alpha \equiv \alpha(g^{-1}x)\). We also have a \(G\) action on the Lie algebra: \(\text{Ad}(g)X = gXg^{-1}\). An “equivariant” form is now one such that
\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\alpha} & \Omega(M) \\
\text{Ad}(g) \downarrow & & \downarrow g \\
\mathfrak{g} & \xrightarrow{\alpha} & \Omega(M)
\end{array}
\]
is a commutative diagram.

We can also define a $G$ action on the original Cartan-model form by

$$(g \cdot \alpha)(X) \overset{\text{def}}{=} g(\alpha(\text{Ad}(g^{-1})X)).$$

Here the $g$ on the RHS is acting on the ordinary $\Omega(M)$ form $\alpha(\text{Ad}(g^{-1})X)$. With this action “equivariant” translates to $g \cdot \alpha = \alpha$, or that the Cartan-model $\alpha$ is invariant.

**Illustration: Atiyah-Bott and the Moment map.** When a $G$-action on $M$ preserves a symplectic form $\omega$, we can use the Cartan model to show that the existence of a $G$-equivariantly closed extension $\omega^\#$ of $\omega$ is equivalent to the existence of an equivariant moment map for the $G$ action$^6$.

The existence of an equivariant moment map needs to pass two cohomological hurdles. Preserving the symplectic form means that

$$\mathcal{L}_a \omega = (d_i a + i_a d)\omega = d(i_a \omega) = 0$$

so $i_a \omega$ is automatically closed, but we need it to be to be trivial in $H^1(M, \mathbb{R})$ for there to exist a globally defined $\phi_a$ such that we have Hamilton’s equation

$$d\phi_a = -i_a \omega = -\omega(X_a, \phantom{a}).$$

Assume that we have passed this hurdle.

Next Jacobi tells us that, for any pair of functions $f, g \in C^\infty(M)$, the Lie bracket of their Hamiltonian vector fields obey

$$[X_f, X_g] = X_{\{f,g\}}.$$  

but (because $f$ and $f + \text{const.}$ give rise to the same $X_f$) this only tell us that the Poisson bracket

$$\{\phi_a, \phi_b\}_{\text{PB}} \overset{\text{def}}{=} X_a \phi_b$$

obeys

$$\{\phi_a, \phi_b\}_{\text{PB}} = f_{ab}^c \phi_c + c_{ab}$$

for constants $c_{ab}$. In this moment-map context “equivariance” requires $c_{ab} = 0$. We can seek to add suitable constants $b_a$ to the $\phi_a$ so as to set $c_{ab} = 0$, but the existence of such $b_a$ requires $c_{ab}$ to be trivial in $H^2(g, \mathbb{R})$.

Assume now that we have an equivariantly closed extension

$$\omega^\# = \omega - \mu^a \varphi_a.$$

---

Then, since $d\mu_a = 0$ in the Cartan model,

$$0 = d_C \omega^# = (d - \mu^a_i a)(\omega - \mu^b \phi_b)$$

$$= -\mu^a (i_a \omega + d\phi_a),$$

As the $\mu_a$ are linearly independent, this means that

$$d\phi_a = -i_a \omega$$

and $\phi_a$ is indeed a suitable hamiltonian for $X_a$.

Further, the requirement that $L_a \omega^# = 0$ reads

$$0 = L_a \omega^# = -f^c_{ab} \mu^b \phi_c + \mu^b L_a \phi_b$$

$$= \mu^b (-f^c_{ab} \phi_c + \{\phi_a, \phi_b\})$$

$$= \mu^b c_{ab}$$

Hence $c_{ab}$ is required to be zero.

Atiyah and Bott use the Weyl model, and write the extension as

$$\omega^# = \omega - D(\theta^a \varphi_a),$$

so their algebra is bit more complicated.

**Universal Thom form:** An interesting example of the generation of equivariant forms is given by the Mathai-Quillen construction of a “universal Thom form.” Let $V$ be a $2N$-dimensional vector space with co-ordinates (the components of the vectors) $\xi^i$ and introduce Grassmann objects $\psi_i$ that are to be thought of as an orthonormal basis of $V$ (so no distinction between upstairs and downstairs components). Let $M_a$ be antisymmetric matrices generating the Lie algebra $\mathfrak{o}(2N)$.

We have the action

$$L_a \xi^i = i_a d\xi^i = -M^{ij}_a \xi^j.$$ 

Consider the Berezin integral

$$\Phi[\xi] = \int d[\psi] \exp \left\{ -\frac{1}{2} |\xi|^2 + i\psi_i d\xi^i + \frac{1}{2} \mu^a \psi_i M^{ij}_a \psi_j \right\}$$

where $\xi = (\xi_1, \ldots, \xi_{2N})$ is a vector. Apply the Cartan differential $d_C = d - \mu^a i_a$ to the exponent

$$S[\psi, \xi] \equiv \left\{ -\frac{1}{2} |\xi|^2 + i\psi_i d\xi^i + \frac{1}{2} \mu^a \psi_i M^{ij}_a \psi_j \right\}.$$ 

We use

$$d_C \xi^i = d\xi^i, \quad d_C(d\xi^i) = \mu^a M^{ij}_a \xi^j, \quad d_C \psi_i = 0,$$

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to get
\[ d_C S = - \sum_i \xi^i \{ d \xi^i + i \mu^a M^{ij}_a \psi_j \} \]
\[ = i \sum_i \xi^i \frac{\partial}{\partial \psi_i} S. \]

Thus
\[ d_C \int d[\psi] \exp \{ S \} = i \sum_i \left( \xi^i \int d[\psi] \frac{\partial}{\partial \psi_i} \exp \{ S \} \right) \]
\[ = 0, \]
as the Berezin integral of a derivative is zero.

Effectively we are using the “equation of motion” of the Grassmann fields \( \psi_i \), and so the Cartan differential is an “on-shell” supersymmetry. We can get an “off-shell” supersymmetry by introducing an auxiliary field \( B_i \) and setting
\[ S[\psi, \xi] \rightarrow S[\psi, \xi, B] = \left\{ -i \xi^i B_i - \frac{1}{2} |B|^2 + i \psi_i d\xi^i + \frac{1}{2} \mu^a \psi_i M^{ij}_a \psi_j \right\} \]
and taking
\[ \delta \xi^i = d\xi^i, \quad \delta (d\xi^i) = \mu^a M^{ij}_a \xi^j, \quad \delta \psi_i = B_i, \quad \delta B_i = \mu^a M^{ij}_a B_j. \]

Then our new \( S \) is given by
\[ S = -\delta \left\{ \psi_i \left( i \xi^i + \frac{1}{2} B^i \right) \right\} \]
As \( \psi_i (i \xi^i + B^i / 2) \) is invariant under rotations, we have \( \delta^2 = 0 \) on it, and so \( \delta S[\psi, \xi, B] = 0 \) — even off-shell.

Following our covariant pull-back recipe from the previous section, we find a genuine closed form on the total space \( E \) of a real even-dimensional vector bundle:
\[ \Phi(E) = - \frac{1}{(2\pi)^N} e^{-\frac{1}{2} \sum_i \xi^2} \int d[\psi] \exp \left\{ i \psi_i \nabla \xi^i + \frac{1}{2} \psi_i R^{ij} \psi_j \right\}. \]
Here
\[ R^{ij} = \frac{1}{2} R^{ij}_{\mu
u} dx^\mu dx^\nu \]
is the \( \mathfrak{o}(2N) \)-valued curvature tensor.

**Thom and Euler:** Consider a vector bundle \( \pi : E \rightarrow M \) with fibre \( F \). Let \( \dim(E) = m \) and \( \dim(M) = n \), so that \( \dim(F) \equiv r = m - n \). If \( \Omega^f(E)_0 \) is the
space of compactly supported $l$-forms. There is now a map $\pi_* : \Omega^l(E)_0 \to \Omega^{l-n}_0$ that is given by integrating along the fibres. For any $\beta \in \Omega^{l-n}(M)_0$ and $\mu \in \Omega^l(E)_0$ it obeys
\[
\int_E \pi^* \beta \wedge \mu = \int_M \beta \wedge \pi_* \mu.
\]
We also have $\pi_* d_E = d_M \pi_*$ and hence the Thom isomorphism
\[
T : H^l(M) \to H^{l+r}_{cv}(E)
\]
where $H^*_{cv}$ is the “compact vertical cohomology” (i.e. all forms are required to have compact support along the fibres). This is implemented by constructing a Thom form $\tau$ in the it Thom class $\subseteq H^r_{cv}(E)$ with $\pi_* \tau = 1$ so that
\[
\int_E \tau \wedge \pi^* \omega = \int_M \omega,
\]
and
\[
T(\alpha) = \tau \wedge \pi^* \alpha.
\]

The point of all this is that we recognise that our $\Phi(E)$ is precisely the $\tau$ for our vector bundle. A surprising fact is that if we use a section $\xi$ to pull-back $\Phi(E)$ from $E$ to $M$ as
\[
i^*_\xi(\Phi) = e(E)
\]
then $e(E)$ is the Euler class. This is obvious if we take $\xi$ to be the zero section, so that $\xi$ and $d\xi$ all become zero and the Grassman integral just outputs Pf$(R)$ — but the class is independent of the section. We can also take any section and consider $\gamma \xi$ with $\gamma \to \infty$ so that the $\xi$ so that the $\exp(-\gamma^2 |\xi|^2/2) i$ causes the integral to localize to the zero of the vector field. When $E$ is the tangent bundle of $M$ this gives a proof of the Poincaré-Hopf theorem.

**Fadeev-Popov and Lagrangian BRS:** Suppose we have a manifold $A$ with coordinates $A$, a compact Lie group $G$, and an action $G : A \to A$ where $G$ acts from the right as $g : A \mapsto A^g$. The group acts on functions as $g : F(A) \mapsto (g \cdot F)(A) = F(A^g)$ with infinitesimal actions given by vector fields $X_\alpha$. Because of the right action we need no minus signs when identifying $e^a X_\alpha$ as the infinitesimal displacement corresponding to $g = 1 + e^a \lambda_a \in G$. We will assume that the $X_\alpha$ are Killing orthonormal.

Consider a function $S(A)$ and measure $d[A]$ that are invariant under $A \mapsto A^g$ and seek to extract from the integral
\[
Z = \int_A d[A] e^{-S(A)}.
\]
the redundant factor of Vol(G).

Suppose that \( G^a, \ a = 1, \ldots \dim(G) \) are functions on \( A \) that intersect each gauge orbit once, and once only. Then we claim that

\[
1 = \int_G d[g] \det|X_a G^b(A^g)| \delta(G^a(A^g)),
\]

where \( d[g] \) is the left- and right-invariant Haar measure. The claim is true because we can always shift \( g \) so that the delta condition is satisfied at \( g = 1 \) where

\[
G^b(A^g) = \epsilon^a X_a G^b(A) + O(\epsilon^2)
\]

and \( d[g] = \prod_{a=1}^{\dim(G)} d\epsilon^a \). Now we insert our expression for unity into our integral for \( Z \). We get

\[
\begin{align*}
Z &= \int_A d[A] \int_G d[g] \det|X_a G^b(A^g)| \delta(G^a(A^g)) e^{-S(A)} \\
&= \int_A \int_G d[A]^-1 d[g] \det|X_a G^b(A)| \delta(G^a(A)) e^{-S(A^-1)} \\
&= \int_A d[A] \int_G d[g] \det|X_a G^b(A)| \delta(G^a(A)) e^{-S(A)} \\
&= \Vol(G) \int_A d[A] \det|X_a G^b(A)| \delta(G^a(A)) e^{-S(A)}.
\end{align*}
\]

In passing to the second line we have shifted the \( A \) variable to \( A^g^{-1} \) in all factors, and in passing from the second to the third we have used the invariance of \( S[A] \) and \( d[A] \). We can now define \( \tilde{Z} \) to be \( Z/\Vol(G) \).

Provided the transversality condition is preserved, we can replace \( G^a(A) \) by \( G^a(A) - \omega^a \) and take a Gaussian average over \( \omega^a \) to get

\[
\tilde{Z} = (2\pi)^{-\dim(G)/2} \det|g_{ab}|^{1/2} \int_A d[A] d[\omega] \det|X_a G^b(A)| \delta(G^a(A) - \omega^a) e^{-\frac{1}{2} g_{ab} \omega_b \omega^b - S(A)}
\]

\[
= (2\pi)^{-\dim(G)/2} \det|g_{ab}|^{1/2} \int_A d[A] \det|X_a G^b(A)| e^{-\frac{1}{2} g_{ab} \omega_b \omega^b + i \omega_a G^a - S(A)}
\]

\[
= (2\pi)^{-\dim(G)} \int_A d[A] d[\omega] d[\theta] d[\bar{\theta}] e^{\theta_b X_a G^b(A) \theta^a - \frac{1}{2} g^{ab} \omega_a \omega_b - i \omega_a G^a - S(A)}.
\]

(Here \( g_{ab} \) is some arbitrary metric, and \( \omega^a \) and \( \omega_a \) are unrelated variables.) The exponent

\[
\tilde{\theta}_b X_a G^b \theta^a - \frac{1}{2} g^{ab} \omega_a \omega_b - i \omega_a G^a - S(A)
\]

\[
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\]
is invariant under $\delta = d_{BRS} = d_K + d_{CE} + \theta^a L_a$ where for any ordinary function $F$ of $A$ we have

$$\delta F = \theta^a (X_a F)$$

(so $\delta S = 0$ because it is $G$-invariant) and the other variables transform as

$$\delta \theta^a = -\frac{1}{2} f^{ab}_{bc} \theta^b \theta^c,$$
$$\delta \bar{\theta}_a = i \omega_a,$$
$$\delta \omega_a = 0.$$ The invariance may be checked directly, but is more easily understood from $\delta^2 = 0$ and

$$-S(A) + \bar{\theta}_b X_a G^b \theta^a - \frac{1}{2} |\omega_a|^2 - i \omega_b G^b = -S(A) + \delta \left\{ \bar{\theta}_a \left( \frac{i g^{ab} \omega_b}{2} - G^a \right) \right\}.$$ 

**Hamiltonian BRS:** We have a phase space $M$ with first-class constraints $\varphi_i(x) = 0$ whose Poisson Brackets obey

$$\{ \varphi_a, \varphi_b \} = f^c_{ab} \varphi_c.$$ 

The easy case has $f^c_{ab}$ being constant. This is what I will do here, but it is perhaps misleading as everything survives when $f^c_{ab}$ is a function of position although the math is harder. The game is played in the extended phase space $\bigwedge g \otimes \bigwedge g^* \otimes C^\infty(M)$. We introduce antighosts $\xi_a$ as generators of $\bigwedge g$ and ghosts $\omega^a$ as generators for $\bigwedge g^*$. Take $\delta$ as Koszul-like

$$\delta \xi_a = \varphi_a$$

and $d$ as Chevally-Eilenberg-like. On functions $F$ we have

$$d F = \omega^a (X_a F)$$

and on antighosts and ghosts ghosts we have

$$d \xi_a = f^c_{ab} \xi_c \omega^b,$$
$$d \omega^a = -\frac{1}{2} f^{ab}_{bc} \omega^b \omega^c.$$ 

Then I claim that $d^2 = \delta^2 = d\delta + \delta d = 0$, so we can define the nilpotent BRS
differential to be \( D = d + \delta \). Let’s verify that \( d^2 \xi_a = 0 \):

\[
\begin{align*}
    d^2 \xi_a &= f^c_{ab} f^e_{cd} \omega^d \omega^b + \frac{1}{2} f^c_{ab} \xi_c f^b_{de} \omega^d \omega^e \\
    &= (f^c_{ab} f^e_{cd} + \frac{1}{2} f^e_{ac} f^b_{db}) \xi_c \omega^d \omega^b \\
    &= \frac{1}{2} \left( [[a, b], d] - [[a, d], b] + [a, [d, b]] \right) \xi_e \omega^d \omega^b \\
    &= -\frac{1}{2} \left( [d, [a, b]] + [b, [d, a]] + [a, [b, d]] \right) \xi_e \omega^d \omega^b \\
    &= 0.
\end{align*}
\]

(Were \( f^e_{ab} \) not constant we would need to differentiate it when we square \( d \), and retaining nilpotence would require the addition of extra \( d \)’s à la Batelin-Vilkovsky.)

For negative \( n \), \( H^n(D) \) is zero. For \( n = 0 \) it is the space of gauge invariant functions on the constraint surface, for positive \( n \) a kind of de-Rham cohomology for functions on the constraint surface, but with \( d \) only differentiating along the gauge equivalent leaves.

**Realization of BRST as a graded Poisson-bracket algebra:** We introduce a graded Poisson bracket \( \{ \quad , \quad \} \) which coincides with the usual Poisson bracket on functions, and is extended to act on the Grassmann objects by setting

\[
\{ \xi_a, \omega^b \} = \{ \omega^b, \xi_a \} = \delta^b_a,
\]

and is a derivation or anti-derivation as appropriate.

In the previous section we had set

\[
\begin{align*}
    Df &= \omega^a X_a f = \omega^a \{ \varphi_a, f \} \\
    D\xi_a &= \varphi_a + f^c_{ab} \xi_c \omega^b \\
    D\omega^a &= -\frac{1}{2} f^a_{bc} \omega^b \omega^c
\end{align*}
\]

Let’s see if we can reproduce these as Poisson brackets with

\[
\Omega \defeq \omega^a \varphi_a - \frac{1}{2} f^c_{ab} \omega^a \omega^b \xi_c.
\]

We have firstly

\[
\{ \Omega, f \} = \omega^a \{ \varphi_a, f \}. \quad \checkmark
\]
Next

\[ \{ \Omega, \xi_d \} = \{ \xi_d, \Omega \}, \quad \text{(bracket is symmetric when both odd)} \]
\[ = \{ \xi_d, \omega^a \varphi_a \} - \{ \xi_d, \omega^a \omega^b \xi_c \frac{1}{2} f_{ab}^{c} \} \]
\[ = \varphi_d - (\delta_a^c \omega^b \xi_d - \omega^a \delta^b_d \xi_c) \frac{1}{2} f_{ab}^{c} \]
\[ = \varphi_d - f_{da}^{c} \omega^a \xi_c \]
\[ = \varphi_d + f_{da}^{c} \xi_c \omega^a, \quad \checkmark \]

and finally

\[ \{ \Omega, \omega^d \} = \{ \omega^d, \omega^a \varphi_a - \frac{1}{2} f_{ab}^{c} \omega^a \omega^b \xi_c \} \]
\[ = -\frac{1}{2} f_{dab} \omega^a \omega^b. \quad \checkmark \]

We see that \( \Omega \) does indeed generate the action of the BRS operator \( D \). Further

\[ \{ \Omega, \Omega \} = 0 \]

by use of the Jacobi identity.

**Action principle, Noether’s theorem**: Let us extract our Poisson algebra from an action

\[ S = \int dt \left( p_a q^a + \xi_d \dot{\omega}^d - H \right). \]

We see that this works if we take \( H = \Omega \) and identify time derivatives with the action of \( D \).

Now if we replace

\[ H \to H + \{ \Omega, \Psi \}, \]

where the gauge fermion has ghost number \(-1\), we get the same equations of motion up to \( D \) cohomology.

To get the Lagrangian Fadeev-Popov for of BRS we take the gauge fermion to be

\[ \Psi = \left\{ \xi_a \left( \frac{ig^{ab} \omega_b}{2} - G^a \right) \right\} \]

where the bosonic \( \omega_a \) has the same transformation rule as the constraint generator \( \varphi_a \).

**Quantum BRS**: 26
To obtain quantum operator corresponding to $d + \delta$ we first take the operators generating the constraints to be $G_a$ with commutator

$$[G_a, G_b] = i f_{abc} G_c.$$  

We have made the factor of $i$ in the structure constants explicit because we need to worry about hermiticity.

We then quantize the formal anticommuting elements by assigning $\omega^a \mapsto c^a$ (ghosts) and $\xi_a \mapsto b_a$ (anti-ghosts) where $c^a$ and $b_a$ are hermitian operators. We make give them the (anti)-commutation relations of conjugate pairs:

$$\{c^a, b^b\} = \delta^a_b, \quad \{c^a, c^b\} = 0, \quad \{b_a, b_b\} = 0$$

We can then set

$$Q = c^a G_a - \frac{i}{2} c^a c^b f_{abc} b_c.$$  

This is hermitian if $f_{ab}^a = 0$ (true if the group is semisimple) and (after some effort) obeys $Q^2 = 0$. The physical states obey $Q|\text{physical}\rangle = 0.$