M. Stone, Jan 2018

Gouy-Maslov phase and Lobachevsky Geometry

The harmonic oscillator time-dependent Schrödinger equation

$$i\partial_t\psi = -\frac{1}{2}\partial_x^2\psi + \frac{1}{2}\omega^2 x^2\psi$$

is (after a replacement $t \to z$) also a one dimensional version of the paraxial wave equation for a graded-index optical fibre. This equation has a "breathing" solution

$$\psi(x,t) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{e^{i\omega t} + Re^{-i\omega t}}} \exp\left\{-\frac{\omega}{2}\left(\frac{1 - R\,e^{-2i\omega t}}{1 + R\,e^{-2i\omega t}}\right)x^2\right\},$$

where the parameter $|R| < 1$. The Gaussian exponent is time-periodic with period one half of of the oscillator period $T = 2\pi/\omega$. In contrast, the prefactor

$$\psi(0,t) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{e^{-i\omega t/2}}{\sqrt{1 + Re^{-2i\omega t}}}$$

accumulates a phase of $-\pi$ in each period T. Because $|R| < 1$, the complex number $1 + Re^{-2i\omega t}$ does not wind about the origin, and so the denominator contributes nothing to the *net* phase increase. It does, however, influence the *rate* at which the phase accumulates: when $R = 0$ the phase increases steadily; when $R \to 1_-$ the accumulation occurs in two rapid shifts of $-\pi/2$ that occur during the twice-per-period moments when the Gaussian wave-packet is very narrow. Depending on the physical context — squeezed quantum oscillator or tightly-focussed Gaussian beam — these phase jumps are known respectively as the the *Maslov phase*[1] or the *Gouy phase*[2].

The breathing solution may be obtained by application of a squeezing operator $S(z)$, $z = e^{i\theta}|z|$, to the oscillator ground state $|0\rangle$. The unitary squeezing operator is given by

$$\begin{aligned} S(z) &\equiv \exp\left\{\tfrac{1}{2}(za^{\dagger 2} - z^* a^2)\right\}, \\ &= \exp\left\{e^{i\theta}\tfrac{1}{2}\tanh|z|\,a^{\dagger 2}\right\}\exp\left\{-\ln\cosh|z|(a^\dagger a + \tfrac{1}{2})\right\}\exp\left\{-e^{-i\theta}\tfrac{1}{2}\tanh|z|\,a^2\right\}, \\ &= \exp\left\{-e^{-i\theta}\tfrac{1}{2}\tanh|z|\,a^2\right\}\exp\left\{+\ln\cosh|z|(a^\dagger a + \tfrac{1}{2})\right\}\exp\left\{e^{i\theta}\tfrac{1}{2}\tanh|z|\,a^{\dagger 2}\right\}. \end{aligned}$$

[1] V. P. Maslov, *Théorie des Perturbations et Méthodes Asymptotiques*, (Dunod, Paris, 1972).

[2] L. G. Gouy, *Sur une proprit nouvelle des ondes lumineuses*, Comptes Rendus Hebdomadaires des Seances de l' Academie des Sciences,110, 1251 (1890)

The oscillator hamiltonian $H = \omega(a^\dagger a + \frac{1}{2})$ provides the time dependence of our wave function, which is

$$\psi(x,t) = \langle x|e^{-iHt}S(z)|0\rangle$$

with $Re^{-2i\omega t} = e^{i\theta}\tanh|z|$. As

$$e^{-iHt}|0\rangle = e^{-i\omega t/2}|0\rangle.$$

and

$$e^{-iHt}S(z)e^{iHt} = S(e^{-2i\omega t}z),$$

we have

$$e^{-iHt}S(z)|0\rangle = e^{-i\omega t/2}S(e^{-2i\omega t}z)|0\rangle.$$

This expression again separates out the periodic factor $S(e^{-2i\omega t}z)|0\rangle$ from the steadily accumulating phase $e^{-i\omega t/2}$. Is there a geometric interpretation of this accumulating phase?

The manifold of squeezed states possesses an inherent Lobachevsky geometry that arises from it being a coset $K = \mathrm{Sp}(2,\mathbb{R})/U(1)$. It has been claimed that the Gouy phase can be considered as the Berry phase associated with this geometry[3]. We can access the geometry by exploiting a Kähler structure that becomes manifest when we work with unnormalized but holomorphic squeezed states. To do this it is convenient to set $\alpha = e^{i\theta}\tanh|z|$ and define

$$|\alpha\rangle = \exp\{\tfrac{1}{2}\alpha a^{\dagger 2}\}|0\rangle.$$

These states are not normalized, and a simple application of MacMahon's master theorem[4] shows that

$$N(\alpha,\alpha^*) \equiv \langle\alpha|\alpha\rangle = \frac{1}{\sqrt{1-|\alpha|^2}}.$$

[3]R. Simon, N. Mukunda, *Bargmann invariant and the geometry of the Güoy effect*, Phys. Rev. Lett. **70** (1993) 880-883.

[4]P A. MacMahon, *Combinatory analysis*, (Cambridge University Press, 1915-16), vol 1, 93-123. His original form of the theorem is equivalen to the statement that

$$\langle 0|\exp\{b_{1i}X_{ij}b_{2j}\}\exp\{b^\dagger_{1k}Y_{kl}b^\dagger_{2l}\}|0\rangle = \frac{1}{\det(1-X^TY)}.$$

The configuration-space wavefunction is given by a theorem of Louck[5] as

$$\langle x|\alpha\rangle = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{1}{\sqrt{(1+\alpha)}} \exp\left\{-\frac{\omega}{2}\left(\frac{1-\alpha}{1+\alpha}\right)x^2\right\},$$

and a simple integration confirms the normalization $N(\alpha,\alpha^*)$. The states

$$\widetilde{|\alpha\rangle} = N^{-1/2}|\alpha\rangle$$

are normalized and coincide with $S(z)|0\rangle$ when $\alpha = e^{i\theta}\tanh|z|$. We wish to find the Berry connection

$$A = i\widetilde{\langle\alpha|d\alpha\rangle} = i\langle 0|S^{-1}(z)dS(z)|0\rangle.$$

The Kähler potential is

$$\ln N = -\frac{1}{2}\ln(1-|\alpha|^2).$$

and from it we obtain

$$\begin{aligned} A &= \frac{i}{2}(\partial_\alpha \ln N d\alpha - \partial_{\alpha^*}\ln N d\alpha^*). \\ &= \frac{i}{4}\frac{(\alpha^* d\alpha - \alpha d\alpha^*)}{1-|\alpha^2|} \end{aligned}$$

and hence the Berry curvature

$$\begin{aligned} F = dA &= (\partial^2_{\alpha^*\alpha}\ln N)\, d\alpha^*\wedge d\alpha \\ &= \frac{i}{2}\frac{d\alpha^*\wedge d\alpha}{(1-|\alpha|^2)^2}. \end{aligned}$$

This should be compared with the area 2-form

$$d[\text{Area}] = \frac{2}{i}\frac{d\alpha^*\wedge d\alpha}{(1-|\alpha|^2)^2}$$

of the Poincaré disc model of Lobachevsky space. We see that the accumulated geometric phase for an adiabatic cyclic evolution is -1/4 the area enclosed in Lobachevsky space.

[5] J. D. Louck, *Extension of the Kibble-Slepian Formula for Hermite Polynomials Using Boson Operator Methods*, Advances in Applied mathematics **2** (1981) 239-249.

The Kähler potential also supplies a quantum metric

$$\begin{aligned} ds^2 &= \|\widetilde{|d\alpha\rangle} - \widetilde{|\alpha\rangle}\widetilde{\langle\alpha|d\alpha\rangle}\|^2 \\ &= (\partial^2_{\alpha^*\alpha} \ln N)\, d\alpha^* \otimes d\alpha \\ &= \frac{1}{2}\frac{d\alpha^* \otimes d\alpha}{(1-|\alpha|^2)^2}, \end{aligned}$$

which is a scaled version of the Poincaré disc metric.

Now we are not so much interested in the adiabatic evolution of the system under a slowly changing hamiltonian, but rather the free cyclic evolution of the state under a time independent Hamiltonian. The accumulating geometric phase is therefore strictly an *Aharonov-Anandan Phase*[6] rather than a Berry phase. In particular our quantum evolution is that of a generalized coherent state $|g\rangle$ under dynamics given by a hamiltonian H that is an element of the Lie algebra of the group — here $\mathrm{Sp}(2,\mathbb{R})$ — of which g is an element. In this case the relevant result is due to Léon Van Hove[7] and says that the coherent state parameter evolves under the classical equations of motion while the quantum state accumulates a phase $e^{i\Phi}$, where

$$\Phi = \int \{i\langle g|dg\rangle - \langle g|H|g\rangle\}\, dt$$

is the classical action evaluated along the classical trajectory. We will call this phase a *Van Hove phase*. It is the sum of the geometric Aharonov-Anandan/Berry phase and a dynamical phase from $\langle g|H|g\rangle$.

In our case

$$\langle g|H|g\rangle \to \langle 0|S^\dagger(ze^{-2i\omega t})\omega(a^\dagger a + \tfrac{1}{2})S(ze^{-2i\omega t})|0\rangle = \frac{\omega}{2}\cosh 2|z| = \omega(\cosh^2|z| - \tfrac{1}{2})$$

and, with $\alpha = |\alpha|e^{i\theta} = e^{i\theta}\tanh|z|$ at fixed $|z|$, we have

$$i\langle g|dg\rangle \to A = -\frac{|\alpha|^2 d\theta}{1-|\alpha|^2} = -\frac{1}{2}(\cosh^2|z| - 1)d\theta$$

[6] Y. Aharonov, J. Anandan, *Phase change during a cyclic quantum evolution*, Phys. Rev. Lett., 58 (1987) 1593-1596.

[7] L. van Hove, *Sur le problème des relations entre les transformations unitaires de la mecanique quantique et les transformations canoniques de la mecanique classique*, Acad. Roy. Belg. Bull. Classe Sci. Mem. **37** (1951) 610-620.

so the accumulated Van Hove phase is

$$\Phi = \int \left\{ \frac{1}{2}(1 - \cosh^2 |z|) d\theta - \omega(\cosh^2 |z| - \tfrac{1}{2}) dt \right\}.$$

The evolution of the coherent state parameter $z = e^{i\theta}|z|$ is $d\theta = -2\omega dt$, so the accumulated phase is

$$\Phi = -\tfrac{1}{2}\omega t,$$

as it should be.

We see that the geometric phase arising from the Lobachevsky geometry can be arbitrarily large, but its contribution is almost completely cancelled by the dynamic phase arsing from the fact that the squeezed "ground state" has large energy expectation value. The optical Gouy/Maslov phases should probably be considered as a *Van Hove phase*, and *not* a Berry phase. We also note that in the geometric phase language the rapid acquisition of the Gouy phase at the nodes of paraxial ray is a gauge dependent statement as it requires our choosing a particular phase convention for the coherent states; similarly the phase acquisition in the oscillator language. Were we to work in momentum space, we would naturally choose phases so that the rapid phase acquisition would occur when the momentum space distribution is narrow and the real space wave-function is at it its broadest extent. The total phase acquisition in a complete period is, however, an invariant.