Kähler Potential and Berry Curvature

Let \(|\psi\rangle = N^{-1/2}(z, \bar{z})|z\rangle\) be a normalized state with \(N^{-1/2}(z, \bar{z})\) a real normalization factor. The notation is intended to indicate that \(|z\rangle\) depends holomorphically on \(z\). Similarly \(\langle z|\) is to depend only on \(\bar{z}\). For example we might have

\[
|\psi\rangle = \frac{1}{\sqrt{1 + |z|^2}} \left( \frac{1}{z} \right), \quad \langle \psi| = \frac{1}{\sqrt{1 + |z|^2}} \left( 1, \bar{z} \right).
\]

From \(N = \langle z|z\rangle\) we deduce that

\[
\partial_z N dz + \partial_{\bar{z}} N \bar{dz} = \langle dz|z\rangle + \langle z|dz\rangle
\]

and read off that

\[
\langle z|dz\rangle = \partial_z N dz,
\]

\[
\langle dz|z\rangle = \partial_{\bar{z}} N d\bar{z}.
\]

We compute the Berry connection as

\[
A \equiv \langle \psi|d\psi\rangle = N^{-1}\langle z|dz\rangle + N^{-1/2}\langle z|d(N^{-1/2})|z\rangle
\]

\[
= \partial_z \ln N dz - \frac{1}{2}(\partial_z \ln N + \partial_{\bar{z}} \ln N)
\]

\[
= \frac{1}{2}(\partial_z \ln N dz - \partial_{\bar{z}} \ln N d\bar{z}).
\]

The Berry Curvature \(F = dA\) is

\[
F = \partial_{zz}^2 \ln N d\bar{z} \wedge dz,
\]

and so appears as a Kähler 2-form \(F = \omega = \bar{\partial} \partial \ln N\).

We define a metric on the space of rays by setting

\[
 ds^2 = \|d\psi \|^2 = \|d\psi - \langle \psi|\psi\rangle d\psi\|^2
\]

\[
= \langle d\psi|d\psi\rangle - \langle d\psi|\psi\rangle \langle \psi|d\psi\rangle.
\]

Here \(dz\) and \(d\bar{z}\) are not 1-forms. They are small changes \(z\) and \(\bar{z}\) respectively, and therefore commute.
We have
\[
\langle d\psi | \psi \rangle \langle \psi | d\psi \rangle = -\frac{1}{4} (\partial_z \ln N dz - \partial_\bar{z} \ln N d\bar{z})^2
\]
and
\[
|d\psi\rangle = -\frac{1}{2} N^{-1/2} (\partial_z \ln N dz + \partial_\bar{z} \ln N d\bar{z}) |z\rangle + N^{-1/2} |dz\rangle
\]
so
\[
\langle d\psi | d\psi \rangle = -\frac{1}{4} (\partial_z \ln N dz + \partial_\bar{z} \ln N d\bar{z})^2 + N^{-1} \langle dz | dz \rangle.
\]
Assembling we find
\[
ds^2 = N^{-1} \langle dz | dz \rangle - (\partial_z \ln N dz)(\partial_\bar{z} \ln N d\bar{z}).
\]
We calculate \( \langle dz | dz \rangle \) from
\[
N^{-1} \langle z | dz \rangle = \partial_z \ln N dz
\]
by varying \( \bar{z} \) while keeping \( z \) fixed. This strategy gives
\[
-(\partial_\bar{z} \ln N d\bar{z})(\partial_z \ln N dz) + N^{-1} \langle dz | dz \rangle = \partial_{zz}^2 \ln N d\bar{z} dz,
\]
so finally
\[
ds^2 = (\partial_{zz}^2 \ln N) d\bar{z} dz.
\]
With multiple \( z \)'s these formulæ generalize to
\[
\omega = (\partial_{zi,\bar{z}j}^2 \ln N) d\bar{z}_i \wedge dz_j,
\]
\[
ds^2 = (\partial_{zi,\bar{z}j}^2 \ln N) d\bar{z}_i \otimes dz_j.
\]
That both the the metric and the closed \((1,1)\)-form \( \omega \) are derived form a Kähler potential \( N \) is a defining characteristic of a Kähler manifold.

**Example:** Let \(|0\rangle\) be the harmonic oscillator ground state and define the squeezed state
\[
|\alpha\rangle = \exp\{\frac{1}{2} \alpha a^2\} |0\rangle
\]
Then, from MacMahon’s master theorem, we have
\[
N(\alpha, \alpha^*) = \frac{1}{\sqrt{1 - |\alpha|^2}}
\]
making
\[ \ln N = -\frac{1}{2} \ln(1 - |\alpha|^2). \]

Then
\[ A = \frac{1}{4} \left( \alpha^* d\alpha - \alpha d\alpha^* \right) \]
\[ dA = \frac{1}{2} \left( d\alpha^* \wedge d\alpha \right) \]

and
\[ ds^2 = \frac{1}{2} \left( d\alpha^* \otimes d\alpha \right) \]

These last two expressions are, respectively, proportional to the area 2-form and metric on the Poincaré disc model of Lobachevsky space.

**Complex and real structures** The matrix
\[ h_{ij} = \partial^2_{z_i z_j} \ln N \]
is Hermitian and can be written as \( h = a + ib \) where \( a \) is real symmetric and \( b \) is real skew-symmetric. If we set \( z_i = x_i + iy_i \) etc. then
\[ \omega = i(dx, dy) \begin{pmatrix} b & a \\ -a & b \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \]

and
\[ ds^2 = (dx, dy) \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} \]

Notice that
\[ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \]

where
\[ J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]
is the matrix representing the complex structure.