M. Stone, Dec 2016

Kähler Potential and Berry Curvature

Let $|\psi\rangle = N^{-1/2}(z,\overline{z})|z\rangle$ be a normalized state with $N^{-1/2}(z,\overline{z})$ a real normalization factor. The notation is intended to indicate that $|z\rangle$ depends holomorphically on z. Similarly $\langle z|$ is to depend only on $\overline{z}$. For example we might have

$$|\psi\rangle = \frac{1}{\sqrt{1+|z|^2}}\begin{pmatrix}1\\ z\end{pmatrix}, \quad \langle\psi| = \frac{1}{\sqrt{1+|z|^2}}\left(1,\overline{z}\right).$$

From $N = \langle z|z\rangle$ we deduce that

$$\partial_z N dz + \partial_{\overline{z}} N d\overline{z} = \langle dz|z\rangle + \langle z|dz\rangle$$

and read off that

$$\begin{aligned}\langle z|dz\rangle &= \partial_z N dz,\\ \langle dz|z\rangle &= \partial_{\overline{z}} N d\overline{z}.\end{aligned}$$

We compute the Berry connection as

$$\begin{aligned}A \equiv \langle\psi|d\psi\rangle &= N^{-1}\langle z|dz\rangle + N^{-1/2}\langle z|d(N^{-1/2})|z\rangle\\ &= \partial_z \ln N dz - \frac{1}{2}(\partial_z \ln N + \partial_{\overline{z}} \ln N)\\ &= \frac{1}{2}(\partial_z \ln N dz - \partial_{\overline{z}} \ln N d\overline{z}).\end{aligned}$$

The Berry Curvature $F = dA$ is

$$F = \partial^2_{\overline{z}z} \ln N d\overline{z} \wedge dz,$$

and so appears as a Kähler 2-form $F = \omega = \bar{\partial}\partial \ln N$.

We define a metric on the space of rays by setting

$$\begin{aligned}ds^2 &= \||d\psi\rangle - |\psi\rangle\langle\psi|d\psi\rangle\|^2\\ &= \langle d\psi|d\psi\rangle - \langle d\psi|\psi\rangle\langle\psi|d\psi\rangle.\end{aligned}$$

Here dz and $d\overline{z}$ are *not* 1-forms. They are small changes z and $\overline{z}$ respectively, and therefore commute.

We have

$$\langle d\psi|\psi\rangle\langle\psi|d\psi\rangle = -\frac{1}{4}(\partial_z \ln N dz - \partial_{\overline{z}} \ln N d\overline{z})^2$$

and

$$|d\psi\rangle = -\frac{1}{2}N^{-1/2}(\partial_z \ln N dz + \partial_{\overline{z}} \ln N d\overline{z})|z\rangle + N^{-1/2}|dz\rangle$$

so

$$\langle d\psi|d\psi\rangle = -\frac{1}{4}(\partial_z \ln N dz + \partial_{\overline{z}} \ln N d\overline{z})^2 + N^{-1}\langle dz|dz\rangle.$$

Assembling we find

$$ds^2 = N^{-1}\langle dz|dz\rangle - (\partial_z \ln N dz)(\partial_{\overline{z}} \ln N d\overline{z}).$$

We calculate $\langle dz|dz\rangle$ from

$$N^{-1}\langle z|dz\rangle = \partial_z \ln N dz$$

by varying $\overline{z}$ while keeping z fixed. This strategy gives

$$-(\partial_{\overline{z}} \ln N d\overline{z})(\partial_z \ln N dz) + N^{-1}\langle dz|dz\rangle = \partial^2_{\overline{z}z} \ln N d\overline{z}\, dz,$$

so finally

$$ds^2 = (\partial^2_{\overline{z}z} \ln N) d\overline{z} dz.$$

With multiple z's these formulæ generalize to

$$\begin{aligned} \omega &= (\partial^2_{\overline{z}_i z_j} \ln N)\, d\overline{z}_i \wedge dz_j, \\ ds^2 &= (\partial^2_{\overline{z}_i z_j} \ln N)\, d\overline{z}_i \otimes dz_j. \end{aligned}$$

That both the the metric and the closed $(1,1)$-form ω are derived form a *Kähler potential* N is a defining characteristic of a Kähler manifold.

Example: Let $|0\rangle$ be the harmonic oscillator ground state and define the squeezed state

$$|\alpha\rangle = \exp\{\tfrac{1}{2}\alpha a^{\dagger 2}\}|0\rangle$$

Then, from MacMahon's master theorem, we have

$$N(\alpha, \alpha^*) = \frac{1}{\sqrt{1 - |\alpha|^2}}$$

making

$$\ln N = -\frac{1}{2}\ln(1-|\alpha|^2).$$

Then

$$\begin{aligned} A &= \frac{1}{4}\frac{(\alpha^* d\alpha - \alpha d\alpha^*)}{1-|\alpha^2|} \\ dA &= \frac{1}{2}\frac{d\alpha^* \wedge d\alpha}{(1-|\alpha|^2)^2}, \end{aligned}$$

and

$$ds^2 = \frac{1}{2}\frac{d\alpha^* \otimes d\alpha}{(1-|\alpha|^2)^2}$$

These last two expressions are, respectively, proportional to the area 2-form and metric on the Poincaré disc model of Lobachevsky space.

Complex and real structures The matrix

$$h_{ij} = \partial^2_{\bar{z}_i z_j} \ln N$$

is Hermitian and can be written as $h = a + ib$ where a is real symetric and b is real skew-symmetric. If we set $z_i = x + iy_i$ *etc.* then

$$\omega = i(dx, dy)\begin{pmatrix} b & a \\ -a & b \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

and

$$ds^2 = (dx, dy)\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$

Notice that

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} b & a \\ -a & b \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} b & a \\ -a & b \end{pmatrix}$$

where

$$J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the matrix representing the complex structure.