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Annoying Golf Ball

Sometimes a golf ball will enter the hole near its lip, dip down while rolling round side of the hole, and then pop out again. We examine this paradoxical motion. We assume that the ball rolls on the rough wall of the hole, but that friction is not enough to damp out any acquired component $s = \boldsymbol{\omega} \cdot \mathbf{n}$ of spin perpendicular to the unit normal $\mathbf{n}$ to the wall at the point P of contact of the ball and the wall. The latter assumption means that these are not quite the usual "rolls without slipping" conditions.

Let the ball have mass M, moment of intertial I, and radius a. The hole has radius $b > a$. Take θ to be the angular position of the point of contact P, and gravity to act in the $-\mathbf{e}_z$ direction.

The angular velocity of the ball and its and center-of-mass velocity are therefore

$$
\begin{aligned}
\boldsymbol{\omega} &= s\mathbf{n} - \frac{b}{a}\dot{\theta}\mathbf{e}_z + \frac{1}{a}\dot{z}(\mathbf{n} \times \mathbf{e}_z), \\
\mathbf{V}_{\mathrm{CM}} &= \dot{z}\mathbf{e}_z + (b-a)\dot{\theta}(\mathbf{n} \times \mathbf{e}_z).
\end{aligned}
$$

Taking time derivatives gives us

$$
\begin{aligned}
\dot{\mathbf{n}} &= -\dot{\theta}(\mathbf{n} \times \mathbf{e}_z), \\
(\dot{\mathbf{n}} \times \mathbf{e}_z) &= \dot{\theta}\mathbf{n}, \\
\dot{\mathbf{V}}_{\mathrm{CM}} &= \ddot{z}\mathbf{e}_z + (b-a)(\ddot{\theta}(\mathbf{n} \times \mathbf{e}_z) + \mathbf{n}\dot{\theta}^2).
\end{aligned}
$$

To avoid worrying about reaction forces, we will take moments about the point of contact P of the ball with the wall. We therefore need the angular momentum of the ball about P.

$$
L_{\mathrm{P}} = I\boldsymbol{\omega} + Ma(\mathbf{n} \times \mathbf{V}_{\mathrm{CM}}).
$$

All the dynamics lies in

$$
\dot{L}_{\mathrm{P}} = Mag(\mathbf{n} \times \mathbf{e}_z).
$$

This is

$$
\begin{aligned}
&\{\dot{s}\mathbf{n} - s\dot{\theta}(\mathbf{n} \times \mathbf{e}_z) - \frac{b}{a}\ddot{\theta}\mathbf{e}_z + \frac{1}{a}\ddot{z}(\mathbf{n} \times \mathbf{e}_z) + \frac{1}{a}\dot{z}\dot{\theta}\mathbf{n}\} + Ma\{\ddot{z}(\mathbf{n} \times \mathbf{e}_z) + (b-a)\ddot{\theta}(\mathbf{n} \times \mathbf{e}_z)\} \\
&\quad = Mga(\mathbf{n} \times \mathbf{e}_z).
\end{aligned}
$$

We read off the coefficients of $\mathbf{e}_z$, $\mathbf{n}$ and $(\mathbf{n}\times\mathbf{e}_z)/a$ to find

$$\begin{aligned}\ddot{\theta} &= 0\\ \dot{s}+\frac{1}{a}\dot{z}\dot{\theta} &= 0\\ \left(M+\frac{I}{a^2}\right)\ddot{z}-\frac{I}{a}s\dot{\theta} &= Mg\end{aligned}$$

if the initial spin s is zero, the first two equations give $\dot{\theta}=\dot{\theta}_0$ constant, and $s=-z\dot{\theta}_0/a$ so that

$$\left(Ma^2+I\right)\ddot{z}+I\dot{\theta}_0^2 z = Mga^2.$$

If $\dot{z}(0)$ is zero $z(t)$ executes SHM in which it dips down a distance $d = 2Mga^2/I\dot{\theta}_0^2$ into the hole before exiting.