M. Stone, May-June 2011, updated Oct 2018

Torsion, Cartan connections, and the Dirac equation

Given a vector field $X = X^\mu \partial_\mu$, the action of the covariant derivative ∇_X on a function f is simply

$$\nabla_X f = X^\mu \partial_\mu f \equiv Xf.$$

Acting on a tangent-space frame-field (a *vielbein*) $\mathbf{e}_a = e^\mu_a \partial_\mu$, it is defined by a choice of functions $\omega^b{}_{a\mu}(x)$ and by setting

$$\nabla_X \mathbf{e}_a = \mathbf{e}_b\, \omega^b{}_{a\mu} X^\mu.$$

The action of ∇_X is now extended to any other tensor object by requiring ∇_X to be both *linear* and a *derivation*. If $Y = Y^a \mathbf{e}_a$, we use both these properties to find

$$\begin{aligned} \nabla_X Y &= (\nabla_X Y^a)\mathbf{e}_a + Y^a(\nabla_X \mathbf{e}_a) \\ &= (X^\mu \partial_\mu Y^a)\mathbf{e}_a + Y^a(\omega^b{}_{a\mu}\mathbf{e}_b X^\mu) \\ &= X^\mu(\partial_\mu Y^a + Y^b \omega^a{}_{b\mu})\mathbf{e}_a. \end{aligned}$$

Thus

$$(\nabla_X Y)^a = X^\mu(\partial_\mu Y^a + Y^b \omega^a{}_{b\mu})$$

is the a-th component of the covariant derivative of the vector field Y with respect to the vector field X.

Because of of their utility in working with spinors the functions $\omega^i{}_{j\mu}(x)$ are often called the *spin-connection* coefficients—although they have no inherent relation with spin.

When we take our frame-field basis $\mathbf{e}_a$ to be the coordinate basis

$$\boldsymbol{\partial}_\mu \stackrel{\text{def}}{=} \frac{\partial}{\partial x^\mu}$$

the connection components $\omega^i{}_{j\mu}$ are traditionally written as $\Gamma^\lambda{}_{\nu\mu}$ and are called *Christoffel symbols*. Thus

$$\nabla_X \boldsymbol{\partial}_\nu = \Gamma^\lambda{}_{\nu\mu} X^\mu\, \boldsymbol{\partial}_\lambda,$$

and, with $Y = Y^\mu \boldsymbol{\partial}_\mu$, we have $\nabla_X Y = (\nabla_X Y)^\nu \boldsymbol{\partial}_\nu$ where

$$(\nabla_X Y)^\nu = X^\mu(\partial_\mu Y^\nu + \Gamma^\nu{}_{\lambda\mu} Y^\lambda).$$

Our index-ordering convention for ${\Gamma^\lambda}_{\nu\mu}$ is that of Misner, Thorne and Wheeler (MTW) and is intended to preserve the correspondence between ${\omega^i}_{j\mu}$ and ${\Gamma^\nu}_{\lambda\mu}$. Many authors interchange the order of the subscripts on ${\Gamma^\nu}_{\lambda\mu}$, but the *torsion tensor* $\mathbf{T}$ is universally defined by

$$\nabla_X Y - \nabla_Y X - [X, Y] = \mathbf{T}(X, Y).$$

Here

$$[X, Y] \stackrel{\text{def}}{=} (X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu)\boldsymbol{\partial}_\mu$$

is the Lie bracket of the vector fields X, and Y. When we write

$$(\mathbf{T}(X, Y))^\nu = {T^\nu}_{\lambda\mu} X^\lambda Y^\mu,$$

our convention gives

$${T^\nu}_{\lambda\mu} = {\Gamma^\nu}_{\mu\lambda} - {\Gamma^\nu}_{\lambda\mu},$$

which looks a bit backwards, but—as said above—the motivation for the index-placement convention is that it is consistent with the usual index placement in the spin-connection coefficients ${\omega^i}_{j\mu}$.

The *curvature* $\mathbf{R}$ defines a map $\mathbf{R}(X, Y) : \mathrm{TM} \to \mathrm{TM}$ given by

$$Z \mapsto \mathbf{R}(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$

In components

$$(\mathbf{R}(X, Y)Z)^\lambda = {R^\lambda}_{\alpha\mu\nu} Z^\alpha X^\mu Y^\nu.$$

Warning: I prefer the notation ∇_X to the apparently more explicit $X^\mu \nabla_\mu$ because in the presence of torsion there is a dangerous ambiguity lurking in the notation ∇_μ. To see this recall that the curvature definition

$$[\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z = \mathbf{R}(X, Y)Z$$

holds even when the connection has torsion. Now take $X \to \partial_\mu$ and $Y \to \partial_\nu$ *i.e.* $X^\lambda = \delta^\lambda_\mu$, $Y^\lambda = \delta^\lambda_\nu$ so that we are tempted to write

$$\nabla_{\partial_\mu} = \delta^\lambda_\mu \nabla_\lambda \stackrel{?}{=} \nabla_\mu, \quad \nabla_{\partial_\nu} = \delta^\lambda_\nu \nabla_\lambda \stackrel{?}{=} \nabla_\nu.$$

Now, remembering that $[\partial_\mu, \partial_\nu] = 0$, our curvature definition gives us

$$([\nabla_\mu, \nabla_\nu]Z)^\lambda \stackrel{?}{=} Z^\alpha {R^\lambda}_{\alpha\mu\nu}.$$

If, however, we include a Christoffel symbol ${\Gamma^\alpha}_{\nu\mu}$ when ∇_μ acts from the left on the index ν on ∇_ν and similarly a ${\Gamma^\alpha}_{\mu\nu}$ when ∇_ν is on the left and acting on the index μ, we end up with

$$([\nabla_\mu, \nabla_\nu]Z)^\lambda \stackrel{?}{=} Z^\alpha {R^\lambda}_{\alpha\mu\nu} - {T^\sigma}_{\mu\nu}\nabla_\sigma Z^\lambda.$$

Which is equation is correct? With the last version, remembering that

$$\nabla_X Y - \nabla_Y X = [X, Y] + T(X, Y),$$

we can use Liebnitz rule to find

$$\begin{aligned}([X^\mu\nabla_\mu, Y^\nu\nabla_\nu]Z)^\lambda &= (\nabla_X Y - \nabla_Y X)^\sigma \nabla_\sigma Z^\lambda + X^\mu Y^\nu (Z^\alpha {R^\lambda}_{\alpha\mu\nu} - {T^\sigma}_{\mu\nu}\nabla_\sigma Z^\lambda) \\ &= Z^\alpha {R^\lambda}_{\alpha\mu\nu} X^\mu Y^\nu + [X, Y]^\sigma \nabla_\sigma Z^\lambda,\end{aligned}$$

which reproduces the curvature definition for general vector fields X, Y. Both formulæ are correct in their own ways. The difference lies in whether we regard the μ on ∇_μ as a shorthand label specifying the basis vector $\boldsymbol{\partial}_\mu$, or as an index on the numerical component of a covector.

Cartan's Structure equations: Élie Cartan regards the array of numbers ${\omega^i}_{j\mu}$ as defining a matrix-valued one-form $\boldsymbol{\omega}$ whose matrix entries are

$${\boldsymbol{\omega}^i}_j = {\omega^i}_{j\mu}\, dx^\mu.$$

With notation we can write

$$\nabla_X \mathbf{e}_a = \mathbf{e}_b\, {\omega^b}_{a\mu} X^\mu = \mathbf{e}_b\, {\boldsymbol{\omega}^b}_a(X).$$

In Cartan's language the curvature becomes the matrix-valued 2-form

$$\mathbf{R} = d\boldsymbol{\omega} + \boldsymbol{\omega} \wedge \boldsymbol{\omega}.$$

Writing out the matrix indices in full this is

$${\mathbf{R}^i}_j = d{\boldsymbol{\omega}^i}_j + {\boldsymbol{\omega}^i}_k \wedge {\boldsymbol{\omega}^k}_j.$$

When our manifold is equiped with a metric we often take the basis vectors $\mathbf{e}_a$ to be orthonormal, so we can raise and lower Roman frame indices without cost. We have not so far made any such assumption of orthonormality — but even were we to do so it remains important to distinguish between

the tangent space TM_p at a point p and its dual space $(\mathrm{TM}_p)^*$. At each point p we define the *co-frame* $\mathbf{e}^{*k} = e^{*k}_\mu\, dx^\mu$ for the co-tangent bundle TM^* to be the basis dual to the $\mathbf{e}_a$ at p. In other words

$$\mathbf{e}^{*a}(\mathbf{e}_b) = \delta^a_b.$$

Cartan uses $\mathbf{e}^{*a}$ to define torsion as the vector-valued 2-form

$$\mathbf{T}^i = d\mathbf{e}^{*i} + \boldsymbol{\omega}^i{}_j \wedge \mathbf{e}^{*j}.$$

Starting from

$$\mathbf{e}^{*i} = e^{*i}_\mu\, dx^\mu,$$

we compute

$$\begin{aligned} d\mathbf{e}^{*i} + \boldsymbol{\omega}^i{}_j \wedge \mathbf{e}^{*j} &= (\partial_\mu e^{*i}_\nu + \omega^i{}_{j\mu} e^{*j}_\nu)\, dx^\mu \wedge dx^\nu \\ &= (\partial_\mu e^{*i}_\nu + \{e^{*i}_\alpha \partial_\mu e^\alpha_j + e^{*i}_\alpha e^\lambda_j \Gamma^\alpha{}_{\lambda\mu}\} e^{*j}_\nu) dx^\mu \wedge dx^\nu \\ &= (\partial_\mu e^{*i}_\nu + e^{*i}_\alpha e^{*j}_\nu \partial_\mu e^\alpha_j + e^{*i}_\alpha \Gamma^\alpha{}_{\nu\mu}) dx^\mu \wedge dx^\nu \\ &= e^{*i}_\alpha \Gamma^\alpha{}_{\nu\mu}\, dx^\mu \wedge dx^\nu \\ &= \tfrac{1}{2} e^{*i}_\alpha T^\alpha{}_{\mu\nu}\, dx^\mu \wedge dx^\nu. \end{aligned}$$

In passing from the third line to the fourth we used $e^{*i}_\alpha e^\alpha_j = \mathbf{e}^{*i}(\mathbf{e}_j) = \delta^i_j$ to deduce

$$\partial_\mu e^{*i}_\nu = -e^{*i}_\alpha (\partial_\mu e^\alpha_j) e^{*j}_\nu.$$

We see that, apart from the trivial greek-to-roman index conversion, Cartan's definition coincides with the universal one.

The two formulæ

$$\begin{aligned} \mathbf{T}^i &= d\mathbf{e}^{*i} + \boldsymbol{\omega}^i{}_j \wedge \mathbf{e}^{*j}, \\ \mathbf{R}^i{}_j &= d\boldsymbol{\omega}^i{}_j + \boldsymbol{\omega}^i{}_k \wedge \boldsymbol{\omega}^k{}_j. \end{aligned}$$

are Cartan's first and second *structure equations*. From them we deduce the first and second *Bianchi identities*

$$\begin{aligned} d\mathbf{T}^i + \boldsymbol{\omega}^i{}_j \wedge \mathbf{T}^j - \mathbf{R}^i{}_k \wedge \mathbf{e}^{*k} &= 0, \\ d\mathbf{R}^i{}_j + \boldsymbol{\omega}^i{}_k \wedge \mathbf{R}^k{}_j - \mathbf{R}^i{}_k \wedge \boldsymbol{\omega}^k{}_j &= 0. \end{aligned}$$

A badly named result: We can also use the co-frame to write down the relation linking the general-basis spin-connection coefficients to the coordinate-basis Christoffel coefficients. From $\mathbf{e}_a = e^\mu_a \partial_\mu$ we compute $\nabla_X \mathbf{e}_a$ in two ways:

$$\mathbf{e}_c\, \omega^c{}_{a\mu} X^\mu = \nabla_X \mathbf{e}_a = X^\mu (\partial_\mu e^\nu_a + e^\lambda_a \Gamma^\nu{}_{\lambda\mu}) \partial_\nu,$$

and use $e^{*a}_\nu = \mathbf{e}^{*a}(\partial_\mu)$ to read off that

$$\omega^a{}_{b\mu} = \mathbf{e}^{*a}(\mathbf{e}_c\,\omega^c{}_{b\mu}) = e^{*a}_\nu(\partial_\mu e^\nu_b + e^\lambda_b \Gamma^\nu{}_{\lambda\mu}).$$

We can also expand out $\mathbf{e}_c = e^\nu_c \partial_\nu$ to write

$$\mathbf{e}_c\,\omega^c{}_{a\mu} X^\mu = X^\mu(\partial_\mu e^\nu_a + e^\lambda_a \Gamma^\nu{}_{\lambda\mu})\partial_\nu$$

as

$$\partial_\mu e^\nu_a + e^\lambda_a \Gamma^\nu{}_{\lambda\mu} - e^\nu_c \omega^c{}_{a\mu} = 0.$$

Similarly, writing ∇_μ as a shorthand for ∇_{∂_μ} so that

$$\nabla_\mu dx^\nu = -\Gamma^\nu{}_{\lambda\mu}\,dx^\lambda, \quad \nabla_\mu \mathbf{e}^{*a} = -\omega^a{}_{b\mu} e^{*b}$$

we find that

$$\partial_\mu e^{*a}_\nu - \Gamma^\lambda{}_{\nu\mu} e^{*a}_\lambda = -\omega^a{}_{b\mu} e^{*b}_\nu.$$

The two results

$$\begin{aligned} \partial_\mu e^{*a}_\nu + \omega^a{}_{b\mu} e^{*b}_\nu - \Gamma^\lambda{}_{\nu\mu} e^{*a}_\lambda &= 0 \\ \partial_\mu e^\nu_a + e^\lambda_a \Gamma^\nu{}_{\lambda\mu} - e^\nu_c \omega^c{}_{a\mu} &= 0, \end{aligned}$$

are forms of what is sometimes called the "tetrad postulate"[1], but this is a bad name. A postulate is something that we are free to accept or reject, but these equations are simply the relation expressing $\omega^a{}_{b\mu}$ in terms of $\Gamma^\lambda{}_{\nu\mu}$ (or *vice versa*) and must always hold.

Note further that neither of these these formulæ require any metric compatability or torsion-free conditions.

Soldering: An interpretation of the "tetrad postulate" comes from considering the vector-valued *solder form*

$$\mathbf{E} = e^{*a}_\mu(\mathbf{e}_a \otimes dx^\mu) = \mathbf{e}_a \otimes \mathbf{e}^{*a} = e^\nu_a(\boldsymbol{\partial}_\nu \otimes \mathbf{e}^{*a}).$$

This provides a map $\mathbf{E} : \mathrm{TM} \to \mathrm{TM}$ and if $\mathbf{x} = x^b \mathbf{e}_b$ we have

$$\mathbf{E}(\mathbf{x}) = \mathbf{e}_a \otimes \mathbf{e}^{*a}(\mathbf{x}) = \mathbf{e}_a x^a = \mathbf{x}.$$

Thus $\mathbf{E}$ is acts as an identity map[2]. In effect, however, $\mathbf{E}$ links two different interpretations of "$\mathbf{x}$": the input $\mathbf{x}$ is a small displacement, starting at p,

[1] Sean. M. Carroll, Space-time and Geometry p487

[2] MTW, p353, use the the notation $\mathbf{d}\mathcal{P}$ for $\mathbf{E}$, where $\mathcal{P}$ denotes a point.

on the curved manifold M, while the output vector $\mathbf{x}$ is an element of the flat vector space that forms the fibre of TM at p. Thus $\mathbf{E}$ maps (or solders) points in a neighbourhood of $p \in$M to vectors near the origin of TM_p. A generic fibre bundle possesses no object like $\mathbf{E}$.

Applying

$$\nabla_\mu \mathbf{e}_a = \mathbf{e}_b\, \omega^b{}_{a\mu}, \quad \nabla_\mu dx^\nu = -\Gamma^\nu{}_{\lambda\mu}\, dx^\lambda$$

to $\mathbf{E} = e^{*a}_\nu (\mathbf{e}_a \otimes dx^\nu)$ we compute

$$\begin{aligned} \nabla_\mu \mathbf{E} &= (\nabla_\mu e^{*a}_\nu)\mathbf{e}_a \otimes dx^\nu + e^{*a}_\nu (\nabla_\mu \mathbf{e}_a) \otimes dx^\nu + e^{*a}_\nu\, \mathbf{e}_a \otimes (\nabla_\mu dx^\nu) \\ &= (\partial_\mu e^{*a}_\nu + \omega^a{}_{b\mu} e^{*b}_\nu - \Gamma^\lambda{}_{\nu\mu} e^{*a}_\lambda)\, \mathbf{e}_a \otimes dx^\nu \end{aligned}$$

As $\mathbf{E}$ is an identity map, the property $\nabla_\mu \mathbf{E} = 0$ is mandated by consistency, and we see that this condition is equivalent to one version of the "tetrad postulate."

Alternatively we may expand $\mathbf{E}$ as

$$\mathbf{E} = e^\nu_a (\boldsymbol{\partial}_\nu \otimes \mathbf{e}^{*a})$$

to get

$$\nabla_\mu \mathbf{E} = (\partial_\mu e^\nu_a - e^\nu_a \omega^a{}_{b\mu} + e^\lambda_a \Gamma^\nu{}_{\lambda\mu})(\boldsymbol{\partial}_\nu \otimes \mathbf{e}^{*a}).$$

Thus

$$\partial_\mu e^\nu_a - e^\nu_a \omega^a{}_{b\mu} + e^\lambda_a \Gamma^\nu{}_{\lambda\mu} = 0,$$

which is the second version of the "tetrad postulate."

Another route to the Bianchi identities: An alternative route to the Bianchi identities that formalizes the dangerous difference between ∇_X and ∇_μ appears in Robert Gowdy's lectures[3]. He defines a new object D_X that acts for example on $T(Y, Z)$ as

$$D_X T(Y, Z) = \nabla_X T(Y, Z) - T(\nabla_X Y, Z) - T(Y, \nabla_X Z).$$

It thus contains"counterterms" that remove the action of X on the vector arguments in T. The subtractions guarantee that $D_X T(Y, Z)$ at a point p is local in that it it depends on X, Y, Z only at p.

An example is

$$\begin{aligned} D_X D_Y f &= \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f \\ &= X(Yf) - (\nabla_X Y) f. \end{aligned}$$

[3] `https://www.people.vcu.edu/~rgowdy/phys691/pdf/rev.pdf`, section 2.6

With the usual derivatives acting on a function

$$\begin{aligned}
[\nabla_X, \nabla_Y]f &= \nabla_X(\nabla_Y f) - \nabla_Y(\nabla_X f) \\
&= \nabla_X(Yf) - \nabla_Y(Xf) \\
&= (\nabla_X Y)f + Y(Xf) - (\nabla_Y X)f - X(Yf) \\
&= (\nabla_X Y - \nabla_Y X - [X,Y])f \\
&= T(X,Y)f
\end{aligned}$$

where $T(X,Y)f = T^\mu(X,Y)\partial_\mu f$. With the new derivative

$$\begin{aligned}
[D_X, D_Y]f &= (XY - \nabla_X Y)f - (YX - \nabla_Y X)f \\
&= -T(X,Y)f. \\
&= -\nabla_{T(X,Y)} f
\end{aligned}$$

Also

$$[D_X, D_Y]Z = R(X,Y)Z - \nabla_{T(X,Y)} Z,$$

which is the $[\nabla_\mu, \nabla_\nu]$ form of the commutator expression, but now evaluated on local vector fields X, Y.

Let X, Y, Z, W be vector fields ands define

$$J[X,Y,Z,W] = ([\nabla_X, [\nabla_Y, \nabla_Z]] + [\nabla_Y, [\nabla_Z, \nabla_X]] + [\nabla_Z, [\nabla_X, \nabla_Y]])W.$$

This should be zero by the Jacobi identity. Expanding out the first term gives

$$\begin{aligned}
& \nabla_X(R(Y,Z)W + \nabla_{[Y,Z]}W) - R(Y,Z)\nabla_X W - \nabla_{[Y,Z]}\nabla_X W \\
= \; & (\nabla_X R(X,Y))W + R(X,[Y,Z])W + \nabla_{[X,[Y,Z]]}W.
\end{aligned}$$

The cyclic sum of the last term is zero, and replacing $\nabla_X R(Y,Z) \to D_X R(Y,Z) + R(\nabla_X Y, Z) + R(Y, \nabla_X Z)$ and cyclic summing gives

$$\begin{aligned}
J &= \{D_X R(Y,Z) + D_Y R(Z,X) + D_Z R(X,Y) \\
& \quad -R(X,T(Y,Z)) - R(Y,T(Z,X)) - R(Z,T(X,Y))\}W
\end{aligned}$$

So

$$\begin{aligned}
0 &= D_X R(Y,Z) + D_Y R(Z,X) + D_Z R(X,Y) \\
& \quad -R(X,T(Y,Z)) - R(Y,T(Z,X)) - R(Z,T(X,Y)),
\end{aligned}$$

which is a form of the second Bianchi. Similarly acting on a function f we have

$$0 = ([\nabla_X, [\nabla_Y, \nabla_Z]] + [\nabla_Y, [\nabla_Z, \nabla_X]] + [\nabla_Z, [\nabla_X, \nabla_Y]])f,$$

which Gowdy says reduces to

$$0 = D_X T(Y,Z) + D_Y T(Z,X) + D_Z T(X,Y) - R(X,Y)Z - R(Y,Z)X - R(Z,X)Y,$$

which is the first Bianchi identity.

The Covariant derivatives of spinors: A version of the "tetrad postulate" that we will need later involves the Dirac operators and gamma matrices. The Dirac operator on a Riemann manifold is

$$\not{D} = \gamma^a e_a^\mu \left(\partial_\mu + \tfrac{1}{2}\sigma^{bc}\,\omega_{bc\mu}\right) = \gamma^a D_a.$$

Here the γ^a are Hermitian matrices with position-independent entries $(\gamma^a)^\alpha{}_\beta$ whose anticommutators are

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}.$$

The

$$\sigma^{ab} = \tfrac{1}{4}[\gamma^a, \gamma^b]$$

are the skew-Hermitian spinor generators of $\mathfrak{so}(N)$ which obey

$$\begin{aligned} [\sigma^{ab}, \sigma^{cd}] &= \delta^{bc}\sigma^{ad} - \delta^{ac}\sigma^{bd} + \delta^{ad}\sigma^{bc} - \delta^{bd}\sigma^{ac}, \\ [\sigma^{ab}, \gamma^c] &= \gamma^a\delta^{cb} - \gamma^b\delta^{ac}. \end{aligned}$$

The operator

$$D_a \equiv D_{\mathbf{e}_a} = e_a^\mu \left(\partial_\mu + \tfrac{1}{2}\sigma^{bc}\,\omega_{bc\mu}\right) = e_a^\mu D_\mu$$

is the covariant derivative acting on the components of a Dirac spinor ψ^α.

If we introduce position-dependent coordinate-frame gamma matrices $\gamma^\mu = e_a^\mu \gamma^a$ that obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$

then the identities that constitute the "tetrad postulate" combined with the metricity condition $\omega_{ab\mu} = -\omega_{ba\mu}$, can be used to show that

$$\partial_\mu \gamma^\nu + \frac{1}{2}[\sigma^{ab}, \gamma^\nu]\omega_{ab\mu} + \gamma^\lambda \Gamma^\nu{}_{\lambda\mu} = 0.$$

This equation can can be interpreted as "$\nabla_\mu \gamma^\nu = 0$" or as the freedom to pass gamma matrices though covariant derivatives provided that suitable connection forms are tacitly or explicitly included. For example an explicit connection is displayed in

$$\not{D}^2 = \gamma^\nu D_\mu \gamma^\nu D_\nu \psi = \gamma^\mu \gamma^\nu (D_\mu D_\nu - \Gamma^\lambda{}_{\mu\nu} D_\lambda)\psi.$$

while an equivalent form the same equation would be written as

$$\gamma^\mu \nabla_\mu \gamma^\nu \nabla_\nu \psi = \gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu \psi.$$

where the ∇'s tacitly contain the spin connection and Christoffel symbols appropriate to the object on which they act.

The Cartan connection: For a manifold equipped with a metric, the *Cartan connection* is a one-form that takes values in the Lie algebra of the affine, orientation-preserving, Euclidean group $\mathrm{E}^+(n)$. This group is a semidirect product $\mathrm{E}^+(n) = \mathrm{T(n)} \rtimes \mathrm{SO}(n)$ of the translation group $\mathrm{T}(n)$ and the rotation group $\mathrm{SO}(n)$. Note that $\mathrm{T}(n)$ is a normal subgroup of of $\mathrm{E}^+(n)$ and that

$$\mathrm{E}^+(n)/\mathrm{T}(n) = \mathrm{SO}(n),$$

while the coset $\mathrm{E}^+(n)/\mathrm{SO}(n)$ is the affine Euclidean n-plane. The short exact sequence

$$1 \to \mathrm{T}(n) \xrightarrow{i} \mathrm{E}^+(n) \xrightarrow{\pi} \mathrm{SO}(n) \to 1$$

right-splits[4] the as a consequence of the semidirect-product structure.

To see how this works, we observe that, after we have chosen an origin, we can write an affine transformation as

$$\mathbf{x} \mapsto \mathbf{x}' = A\mathbf{x} + \mathbf{b}.$$

This is not itself a linear transformation, but it can be expressed in terms of one by using homogeneous coordinates:

$$\begin{pmatrix} \mathbf{x}' \\ 1 \end{pmatrix} = \begin{pmatrix} A & \mathbf{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}.$$

[4] *I.e.* there is a group morphism $r : \mathrm{SO}(n) \to \mathrm{E}^+(n)$ such that $\pi \circ r = \mathrm{Id}$. The exact sequence does not *left-split* as $\mathrm{SO}(n)$ is not a normal subgroup of $\mathrm{E}^+(n)$. Consequently a map $l : \mathrm{E}^+(n) \to \mathrm{T}(n)$ such that $i \circ l = \mathrm{Id}$ cannot be a group morphism.

We can make the semidirect-product structure "$G = NH$" clear by factorizing.

$$\begin{pmatrix} A & \mathbf{b} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \mathbf{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Products are then

$$\begin{pmatrix} A_2 & \mathbf{b}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 & \mathbf{b}_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_2 A_1 & A_{2\mathbf{1}} + \mathbf{b}_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \mathbf{b}_2 + A_2 \mathbf{b}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_2 A_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is of the semi-direct product form $(n_2, h_2) * (n_1, h_i) = (n_2(h_2 n_1 h_2^{-1}), h_2 h_1)$ because

$$\begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{I} & \mathbf{b}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \mathbb{I} & A_2 \mathbf{b}_1 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} \mathbb{I} & {}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbb{I} & A_2 \mathbf{b}_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \mathbf{b}_2 + A_2 \mathbf{b}_1 \\ 0 & 1 \end{pmatrix}.$$

The Lie algebra of $\mathrm{E}^+(n)$ is generated by the $(n+1)$-by-$(n+1)$ matrices

$$\begin{aligned} \sigma_{ij} &= e_{ij} - e_{ji}, \quad 1 \le i < j \le n \\ \tau_i &= e_{i,n+1}, \quad 1 \le i \le n. \end{aligned}$$

Here e_{ij} is the matrix with unity in the i, j entry and zero elsewhere. From $e_{ij} e_{kl} = \delta_{jk} e_{il}$ we find that

$$[\tau_i, \tau_j] = 0, \quad [\sigma_{ij}, \tau_k] = \delta_{jk} \tau_i - \delta_{ik} \tau_j.$$

The commutators of the σ_{ij} coincide with those of the usual rotation group generators:

$$[\sigma_{ij}, \sigma_{lm}] = \delta_{jl} \sigma_{im} - \delta_{jm} \sigma_{il} + \delta_{im} \sigma_{jl} - \delta_{il} \sigma_{jm}.$$

The *Cartan-connection* is the Lie-algebra-valued one-form

$$\boldsymbol{\eta} = \mathbf{e}^{*i} \tau_i + \frac{1}{2} \boldsymbol{\omega}^{ij} \sigma_{ij}.$$

Here $\mathbf{e}^{*i} = e^i_\mu dx^\mu$ is playing the same rôle that it does in the solder-form: identifying displacements δx^μ on M with vectors in TM. The curvature of this one-form is (after some grinding with the Lie algebra commutators)

$$\begin{aligned} \mathbf{F} &= d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta} \\ &= \frac{1}{2} \sigma_{ij} (d\boldsymbol{\omega}^{ij} + \boldsymbol{\omega}^i{}_m \wedge \boldsymbol{\omega}^{mj}) + \tau_i (d\mathbf{e}^{*i} + \boldsymbol{\omega}^i{}_j \wedge \mathbf{e}^{*j}) \\ &= \frac{1}{2} \sigma_{ij} \mathbf{R}^{ij} + \tau_i \mathbf{T}^i. \end{aligned}$$

The torsion thus manifests itself as the translation component of the affine curvature. The first and second Bianchi identities

$$\begin{aligned} d\mathbf{T}^i + \boldsymbol{\omega}^i{}_j \wedge \mathbf{T}^j - \mathbf{R}^i{}_j \wedge \mathbf{e}^{*j} &= 0, \\ d\mathbf{R}^i{}_j + \boldsymbol{\omega}^i{}_m \wedge \mathbf{R}^m{}_j - \mathbf{R}^i{}_m \wedge \boldsymbol{\omega}^m{}_j &= 0, \end{aligned}$$

are now the translation and rotation components respectively of the single affine Bianchi identity

$$d\mathbf{F} + \boldsymbol{\eta} \wedge \mathbf{F} - \mathbf{F} \wedge \boldsymbol{\eta} = 0.$$

Parallel transport via the affine connection consists of a sequence of infinitesimal affine transformations along the lines of

$$(\mathbf{b}_3, A_3) * (\mathbf{b}_2, A_2) * (\mathbf{b}_1, A_1) = (\mathbf{b}_3 + A_3\mathbf{b}_2 + A_3A_2\mathbf{b}_1, A_3A_2A_1).$$

The translation part of this product motivates Cartan's definition of the *development* of a curve $\gamma^\mu(t)$ in the base space M into a curve $\mathbf{x}(t)$ lying in the affine plane tangent to an initial point $\gamma(0) \in M$. He solves

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{n}_i(t) e^{*i}_\mu(\gamma(t))\dot{\gamma}^\mu(t) \\ \dot{\mathbf{n}}_i(t) &= \mathbf{n}_j(t)\omega^j{}_{i\mu}(\gamma(t))\dot{\gamma}^\mu(t), \end{aligned}$$

with initial conditions $\mathbf{x}(0) = 0$, $\mathbf{n}_i(0) = \mathbf{e}_i$. Note that this is the <u>inverse</u> of the usual parallel transport. There we carry a element of the fibre over the initial point to a point in the fibre over the current point. The "development" construction is mapping the contact point and tangent in the fibre above $\gamma(t)$ back to the fibre over $\gamma(0)$, the points $\mathbf{x}(t) + \delta\mathbf{x}(t) = A(t)\delta\mathbf{x}(0) + \underline{(}t)$ being the image of the neighbourhood of the contact point in tangent space above $\gamma(t)$ under the *inverse* of the parallel-transport map.

The idea is that we "roll" the initial tangent plane over the manifold The developed curve is the path in the plane of the point of contact of the plane with M. In the dislocation/disclination interpretation, the affine tangent plane is the ideal perfect crystal, and the developed path $\mathbf{x}(t)$ is the lifted image in the perfect crystal of the walk $\gamma(t)$ in the imperfect crystal. When we have a closed path with $\gamma(t_0) = \gamma(0)$, then $\mathbf{x}(t_0)$ is the Burgers vector.

The Levi-Civita connection is a "Berry" Connection: The original motivation for preferring the torsion-free Levi-Civita connection over any other metric-compatible connection came from thinking of the n-Manifold

M as embedded (or immersed) isometrically in some Euclidean space $\mathbb{E}^{n+m}$. At each point $p \in M$ we decompose

$$\mathbb{E}^{n+m} = T_p(M) \oplus N_p(M).$$

The frame $\mathbf{e}_a$, $a = 1, \ldots, n$ is now a field of unit tangent vectors to M, and the natural connection is the "Berry" connection that sets

$$\nabla_a \mathbf{e}_b \equiv \mathbf{e}_c \omega^c{}_{ba} = \sum_{c=1}^{n} \mathbf{e}_c \sum_{i,j=1}^{n+m} e_c^j (e_a^i \partial_i e_b^j) = P(\partial_a \mathbf{e}_b).$$

Here the derivatives are computed in $\mathbb{E}^{n+m}$, and P denotes the orthogonal projection to T_p along N_p. Now

$$\begin{aligned} T(\mathbf{e}_a, \mathbf{e}_b) &= \nabla_a \mathbf{e}_b - \nabla_b \mathbf{e}_a - [\mathbf{e}_a, \mathbf{e}_b] \\ &= P(\partial_a \mathbf{e}_b) - P(\partial_b \mathbf{e}_a)) - (\partial_a \mathbf{e}_b - \partial_b \mathbf{e}_a) \\ &= P(\partial_a \mathbf{e}_b - \partial_b \mathbf{e}_a) - (\partial_a \mathbf{e}_b - \partial_b \mathbf{e}_a). \end{aligned}$$

Although the individual $\partial_a \mathbf{e}_b$ and $\partial_b \mathbf{e}_a$ do not lie in T_p, their difference, the Lie bracket of $\mathbf{e}_a$ and $\mathbf{e}_b$, does. The P on the first term in the last line can be dropped therefore, and the two terms cancel. Thus $T(\mathbf{e}_a, \mathbf{e}_b) = 0$. (The Lie bracket can be computed either as a bracket of fields defined in $\mathbb{E}^{n+m}$ or as fields in M.)

Although the Levi-Civita connection is motivated as an *extrinsic* construction —*i.e.* one that depends on the embedding — it is a remarkable fact (Gauss' *Theorema Egregium*) that it may also be extracted from purely *intrinsic* metric data *via* the Levi-Civita formula for the Christoffel symbols.

Second fundamental form: Let $P^\perp$ denote the orthogonal projection to N_p along T_p. The fact that $\partial_X Y - \partial_Y X = [X, Y]$ lies in $T_p(M)$ then shows that

$$N(X, Y) \stackrel{\text{def}}{=} P^\perp(\partial_X Y) = N(Y, X)$$

is a symmetric map $N : T_p(M) \times T_p(M) \to N_p(M)$. It is the *second fundamental form* of the embedded surface.

Group contraction and the Nieh-Yan identity[5]: We can regard our affine group as the result of a group contraction

$$\mathrm{SO}(n+1) \to \mathrm{E}^+(n) = \mathrm{T}(n) \rtimes \mathrm{SO}(n)$$

[5]Note added Oct 2018.

obtained by setting the generators $\sigma_{i,n+1}/l = \tau_i$, and then taking the the radius $l \to \infty$. If we stay near the north pole where $x^{n+1} = 1$ a large radius-l n-sphere S^n is indistinguishable from the affine n-plane $\mathbb{E}^n$:

$$S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n) \to \mathbb{E}^n.$$

If we keep l finite then the algebra $\mathfrak{so}(n+1)$ decomposes as

$$[\sigma_{ij}, \sigma_{lm}] = \delta_{jl}\sigma_{im} - \delta_{jm}\sigma_{il} + \delta_{im}\sigma_{jl} - \delta_{il}\sigma_{jm}, \quad i, j = 1, \ldots, n.$$

$$[\sigma_{ij}, \tau_k] = \delta_{jk}\tau_i - \delta_{ik}\tau_j,$$

but

$$[\tau_i, \tau_j] = -\frac{1}{l^2}\sigma_{ij}.$$

Restricting now to a four-dimensional space, we introduce a less-than-general SO(5) connection by requiring $\boldsymbol{\omega}^{i5}$ to be $\mathbf{e}^{*i}/l$, with $\mathbf{e}^{*i}$ playing its soldering rôle. Then

$$\boldsymbol{\eta} = \frac{1}{l}\sigma_{i5}\mathbf{e}^{*i} + \frac{1}{2}\sigma_{ij}\boldsymbol{\omega}^{ij} = \tau_i\mathbf{e}^{*i} + \frac{1}{2}\sigma_{ij}\boldsymbol{\omega}^{ij}$$

reduces to Cartan's affine group connection as l becomes large. For finite l, however, the SO(5) curvature form has an extra term

$$\begin{aligned} \mathbf{F} = d\boldsymbol{\eta} + \boldsymbol{\eta} \wedge \boldsymbol{\eta} &= \frac{1}{2}\sigma_{ij}(\mathbf{R}^{ij} - \frac{1}{l^2}\mathbf{e}^{*i} \wedge \mathbf{e}^{*j}) + \tau_i\mathbf{T}^i \\ &= \frac{1}{2}\sigma_{ij}(\mathbf{R}^{ij} - \frac{1}{l^2}\mathbf{e}^{*i} \wedge \mathbf{e}^{*j}) + \frac{1}{l}\sigma_{i5}\mathbf{T}^i \end{aligned}$$

Put a positive definite trace operation on $\mathfrak{so}(n+1)$ so that, for $i < j$, $l < m$

$$\mathrm{tr}\,\{\sigma_{ij}\sigma_{lm}\} = 2\delta_{il}\delta_{jm},$$

and

$$\mathrm{tr}\,\left\{\tfrac{1}{2}\sigma_{ij}R^{ij}\tfrac{1}{2}\sigma_{kl}R^{kl}\right\} = R^{ij}R_{ij}.$$

As a consequence, the Chern-Simons identity

$$\mathrm{tr}\,\{\mathbf{F}^2\} = d\,\mathrm{tr}\{\boldsymbol{\eta}d\boldsymbol{\eta} + \tfrac{2}{3}\boldsymbol{\eta}^3\}$$

decomposes as

$$\mathbf{R}^{ij} \wedge \mathbf{R}_{ij} + \frac{2}{l^2}\left(\mathbf{T}^i \wedge \mathbf{T}_i - \mathbf{e}^{*i} \wedge \mathbf{e}^{*j} \wedge \mathbf{R}_{ij}\right) = d\,\mathrm{tr}\{\boldsymbol{\omega}d\boldsymbol{\omega} + \tfrac{2}{3}\boldsymbol{\omega}^3\} + \frac{2}{l^2}d\{\mathbf{e}^{*i} \wedge \mathbf{T}^i\}$$

The parameter l is arbitrary, so we must have

$$\mathbf{N} \stackrel{\text{def}}{=} \mathbf{T}^i \wedge \mathbf{T}_i - \mathbf{e}^{*i} \wedge \mathbf{e}^{*j} \wedge \mathbf{R}_{ij} = d\{\mathbf{e}^{*i} \wedge \mathbf{T}_i\}, \quad (\star).$$

This is the Nieh-Yan identity[6]. Although motivated by the group contraction, the identity $(\star)$ is easily verified by using the first Bianchi identity to directly evaluate the four-form $d(\mathbf{e}^{*i} \wedge \mathbf{T}^i)$.

We can write the Nieh-Yan 4-form in coordinate-index notation as

$$\mathbf{N} = \frac{1}{4}\left\{\frac{\epsilon^{\alpha\beta\gamma\delta}}{\sqrt{g}}(\delta_{ab}T^a{}_{\alpha\beta}T^b{}_{\gamma\delta} - 2R_{\alpha\beta\gamma\delta})\right\}\sqrt{g}\,dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4.$$

Different normalizations are also used.

Although there is no globally defined co-frame $\mathbf{e}^{*a}$ unless the manifold M is parallelizable, the expression $\mathbf{e}^{*i} \wedge \mathbf{T}_i$ is a globally defined. Consequently $\mathbf{N}$ is an exact four-form and $\int_M \mathbf{N} = 0$ for all closed, orientable, smooth manifolds.

Nieh-Yan "topological invariants": Despite $\mathbf{N}$ being an exact form, there are claims in the literature[7] that there are spaces with non-zero values of $\int_M \mathbf{N}$. The claimed examples are spaces parameterized by $S^3 \times \mathbb{R}^+$ and exploit the fact that the three sphere is parallelizable. Let is use Euler-angles θ, ϕ, ψ as coordinates on S^3 and define an orthonormal co-frame

$$\mathbf{e}^{*4} = h(r)dr, \quad \mathbf{e}^{*i} = f(r)\Omega^i_L, \quad i = 1, 2, 3,$$

where

$$\begin{aligned}
\Omega^1_L &= \sin\psi d\theta - \sin\theta\cos\psi d\phi, \\
\Omega^2_L &= \cos\psi d\theta + \sin\theta\sin\psi d\phi, \\
\Omega^3_L &= d\psi + \cos\theta d\phi,
\end{aligned}$$

are $(-2\times)$ the left-invariant Maurer-Cartan forms on S^3 considered as the group manifold of SU(2). These forms obey $d\Omega^3_L = -\Omega^1 \wedge \Omega^2$ *etc.* Taking them to be orthonormal leads to the metric

$$ds^2 \equiv \sum_{a=1}^{4} \mathbf{e}^{*a} \otimes \mathbf{e}^{*a}$$

[6]H. T. Nieh, M. L. Yan, *An identity in Riemann-Cartan geometry*, Journal of Mathematical Physics **23** 373-374 (1982); doi: 10.1063/1.525379.

[7]For example: Osvaldo Chandia, Jorge Zanelli *Topological Invariants, Instantons and Chiral Anomaly on Spaces with Torsion*, arXiv:hep-th/9702025.

$$
\begin{aligned}
&= h^2 dr^2 + f^2(d\theta^2 + d\phi^2 + d\psi^2 + 2\cos\theta d\varphi^* \psi) \\
&= h^2 dr^2 + 4f^2 dS^2.
\end{aligned}
$$

Here dS^2 is the usual metric on the unit three-sphere in Euler-angle coordinates. A related result is that

$$
\mathbf{e}^{*1} \wedge \mathbf{e}^{*2} \wedge \mathbf{e}^{*3} \wedge \mathbf{e}^{*4} = 8hf^3 d(\mathrm{Vol}[S^3])dr.
$$

Let us now declare that the associated dual TM frame $\mathbf{e}_i$ to be teleparallel. In this frame then the connection form is zero and the torsion two form becomes $\mathbf{T}^a = d\mathbf{e}^{*a}$. Explicitly

$$
\begin{aligned}
\mathbf{T}^1 &= \frac{1}{f}\left(\frac{df}{dr} dr \wedge \mathbf{e}^{*1} - \mathbf{e}^{*2} \wedge \mathbf{e}^{*3}\right), \\
\mathbf{T}^2 &= \frac{1}{f}\left(\frac{df}{dr} dr \wedge \mathbf{e}^{*2} - \mathbf{e}^{*3} \wedge \mathbf{e}^{*1}\right), \\
\mathbf{T}^3 &= \frac{1}{f}\left(\frac{df}{dr} dr \wedge \mathbf{e}^{*3} - \mathbf{e}^{*1} \wedge \mathbf{e}^{*2}\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbf{N} &\equiv \mathbf{T}^a \wedge \mathbf{T}_a \\
&= -6\frac{1}{f^2}\frac{df}{dr} dr \wedge \mathbf{e}^{*1} \wedge \mathbf{e}^{*2} \wedge \mathbf{e}^{*3} \\
&= -48 f \frac{df}{dr} dr\, d(\mathrm{Vol}[S^3]),
\end{aligned}
$$

and

$$
\mathbf{e}^{*a} \wedge \mathbf{T}_a = -\frac{3}{f}\mathbf{e}^{*1} \wedge \mathbf{e}^{*2} \wedge \mathbf{e}^{*3} = -24 f^2 d(\mathrm{Vol}[S^3]).
$$

This is similar to equation 3.12 in Obukhov *et al.*[8]

It must clear that the associated dual frame $\mathbf{e}_a$ cannot be smoothly continued to the origin $r = 0$ without become multivalued. Chandia and Zanelli seem to ignore this problem by choosing $f = 1$. This choice makes $\mathbf{N} \equiv 0$, but $\int \mathbf{e}^{*a} \wedge \mathbf{T}_a$ takes the same non-zero value on any 3-surface enclosing the origin. In this case the set of points at $r = 0$ has a non-zero surface area and

[8] Yuri N. Obukhov, Eckehard W. Mielke, Jan Budczies, Friedrich W. Hehl, *On the chiral anomaly in non-Riemannian spacetimes* arXiv:gr-qc/9702011

is a three sphere. Effectively their manifold has a spherical hole removed and possesses boundaries at both $r = 0$ and $r = \infty$. Including the two boundary contributions allows Stokes' theorem to be consistent with $\int_M \mathbf{N} = 0$. Chandia and Zanelli and also Obukhov *et al.* consider an alternative by taking $f \to 0$ at $r = 0$ sufficiently rapidly that the multivaluedness is avoided. Now, however, the $\mathbf{e}_a$ fail to constitute a basis for the tangent space at $r = 0$. Further, as a result of this scaling to zero, the rapid shrinkage of the 3-sphere areas means that the Riemann manifold has an infinitely sharp spike singularity at the origin, and so is not smooth.

Contorsion: Let $\mathring{\Gamma}^{\lambda}{}_{\mu\nu}$ denote the Levi-Civita (torsionless) connection, and $\Gamma^{\lambda}{}_{\mu\nu}$ any metric compatible connection with torsion

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\mu\nu}.$$

Then the difference

$$K^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \mathring{\Gamma}^{\lambda}{}_{\mu\nu}$$

is a tensor obeying

$$K^{\lambda}{}_{\mu\nu} - K^{\lambda}{}_{\nu\mu} = -T^{\lambda}{}_{\mu\nu}.$$

Because both $\Gamma^{\lambda}{}_{\nu\mu}$ and $\mathring{\Gamma}^{\lambda}{}_{\nu\mu}$ are metric compatible, we must have

$$K^{\lambda}{}_{\alpha\mu} g_{\lambda\beta} + K^{\lambda}{}_{\beta\mu} g_{\alpha\lambda} = 0.$$

In other words $K_{\lambda\mu\nu}$ must be antisymmetric in its first two indices, while $T_{\lambda\mu\nu}$ is antisymmetric in its last two indices. Thus, after some algebra, we find

$$K_{\lambda\mu\nu} = -\frac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\nu\lambda} - T_{\nu\lambda\mu}).$$

Palatini: The 4-d Einstein-Hilbert action in vierbein notation is

$$S = \frac{1}{2}\int_M \epsilon_{ijkl}\mathbf{R}^{ij} \wedge \mathbf{e}^{*k} \wedge \mathbf{e}^{*l}$$

Let us compute the variation

$$\delta S = \delta\left(\frac{1}{2}\int_M \epsilon_{ijkl}\mathbf{R}^{ij} \wedge \mathbf{e}^{*k} \wedge \mathbf{e}^{*l}\right)$$

under a Palatini variation $\boldsymbol{\omega}^{ab} \to \boldsymbol{\omega}^{ab} + \delta\boldsymbol{\omega}^{ab}$. We have that

$$\begin{aligned}
\delta\mathbf{R}^{ab} &= d\delta\boldsymbol{\omega}^{ab} + \delta\boldsymbol{\omega}^{am} \wedge \boldsymbol{\omega}_m{}^b + \boldsymbol{\omega}^a{}_m \wedge \delta\boldsymbol{\omega}^{mb} \\
&= d\delta\boldsymbol{\omega}^{ab} + \boldsymbol{\omega}^a{}_m \wedge \delta\boldsymbol{\omega}^{mb} + \boldsymbol{\omega}^b{}_m \wedge \delta\boldsymbol{\omega}^{am} \\
&= D\delta\boldsymbol{\omega}^{ab}
\end{aligned}$$

where D is a covariant exterior derivative. We can define this on any SO(n) tensor-valued form, with the tensor living in the internal fibre space. For example on a $(1,1)$ p-form $\boldsymbol{\beta}^i{}_j$ we set

$$D\boldsymbol{\beta}^i{}_j \equiv d\boldsymbol{\beta}^i{}_j + \boldsymbol{\omega}^i{}_m \wedge \boldsymbol{\beta}^m{}_j - \boldsymbol{\omega}^m{}_j \wedge \boldsymbol{\beta}^i{}_m.$$

Although the Euclidean metric for the fibre-space roman indices means that we can raise or lower them without penalty, the minus sign when applying the connection to the covariant index makes it appear that we must keep track of upstairs/downstairs. Observe, however, we can write the above definition as

$$D\boldsymbol{\beta}^{ij} = d\boldsymbol{\beta}^{ij} + \boldsymbol{\omega}^i{}_m \wedge \boldsymbol{\beta}^{mj} + \boldsymbol{\omega}^j{}_m \wedge \boldsymbol{\beta}^{im},$$

because the antisymmety of $\boldsymbol{\omega}_{ij} \overset{\text{numerical}}{=} \boldsymbol{\omega}^i{}_j$ ensures term-by-term numerical identity. We do need to respect the index placement on $\boldsymbol{\omega}^i{}_j$, though. We must also always wedge the connection one-form in from the left.

I believe that D is an antiderivation (since ∇ is a derivation). Also

$$D\epsilon_{abcd} \equiv -\boldsymbol{\omega}^m{}_a\epsilon_{mbcd} - \boldsymbol{\omega}^m{}_b\epsilon_{amcd} - \boldsymbol{\omega}^m{}_c\epsilon_{abmd} - \boldsymbol{\omega}^m{}_d\epsilon_{abcm} = 0,$$

as there is no way to distribute the index values $1,2,3,4$ without having two identical indices on the Levi-Civita. Thus

$$\begin{aligned} d(\epsilon_{abcd}\delta\boldsymbol{\omega}^{ab} \wedge \mathbf{e}^{*c} \wedge \mathbf{e}^{*d}) \quad = \quad & \epsilon_{abcd}\,(D\delta\boldsymbol{\omega}^{ab}) \wedge \mathbf{e}^{*c} \wedge \mathbf{e}^{*d} \\ & -\epsilon_{abcd}\,\delta\boldsymbol{\omega}^{ab} \wedge (D\mathbf{e}^c) \wedge \mathbf{e}^d \\ & +\epsilon_{abcd}\,\delta\boldsymbol{\omega}^{ab} \wedge \mathbf{e}^{*c} \wedge (D\mathbf{e}^{*d}) \end{aligned}$$

The object being "d"-ed on the left is an invariant (SO(n) singlet), so takes the usual d. Now, using Stokes and discarding the boundary term, we have

$$\delta S = \int_M \epsilon_{abcd}\,\delta\omega^{ab}\,(D\mathbf{e}^{*c}) \wedge \mathbf{e}^{*d}.$$

Since

$$D\mathbf{e}^{*c} \equiv d\mathbf{e}^{*c} + \boldsymbol{\omega}^c{}_m \wedge \mathbf{e}^{*m} = \mathbf{T}^c,$$

the Palatini variation implies that $\mathbf{T}^c = 0$. This is no longer so when the action includes fields with intrinsic spin.

Energy-Momentum tensor: The effective action S (what is left after we integrate out all matter fields) is a functional of the metric, or equivalently,

of the frame modulo local rotations. Hilbert defines his energy-momentum tensor $T_{\mu\nu}$ by

$$\begin{aligned}\delta S_{\text{eff}} &= \frac{1}{2}\int d^n x\sqrt{g}\, T_{\mu\nu}\, \delta g^{\mu\nu} \\ &= -\frac{1}{2}\int d^n x\sqrt{g}\, T^{\mu\nu}\, \delta g_{\mu\nu}\end{aligned}$$

We can also define a tensor T_{cb} by varying the frame:

$$\begin{aligned}\delta S_{\text{eff}} &= \int d^n x\sqrt{g}\left(\frac{\delta S}{\delta e^{\mu}_{a}}\right)\delta e^{\mu}_{a} \\ &\equiv \int d^n x\sqrt{g}\left(T_{cb}\eta^{ca}e^{*b}_{\mu}\right)\delta e^{\mu}_{a}.\end{aligned}$$

As defined, there is no immediate reason for T_{cb} to be symmetric. However, the action functional S should be invariant under an infinitesimal local rotation $\delta e^{\mu}_{a} = e^{\mu}_{b}\theta^{b}{}_{a}$, and so

$$\begin{aligned}0 &= \delta S_{\text{eff}} \\ &= \int d^n x\sqrt{g}\, T_{cb}\, \eta^{ca}e^{*b}_{\mu}e^{\mu}_{d}\theta^{d}{}_{a} \\ &= \int d^n x\sqrt{g}\, T_{cb}\, \eta^{ca}\theta^{b}{}_{a} \\ &= \int d^n x\sqrt{g}\, T_{cb}\, \theta^{bc}.\end{aligned}$$

Since θ^{bc} is an arbitrary skew symmetric matrix, we see that $T_{bc} = T_{cb}$. Accepting this as true, we can now set

$$\begin{aligned}\delta S_{\text{eff}} &= \frac{1}{2}\int d^n x\sqrt{g}\, T_{cb}\left(\eta^{ca}e^{*b}_{\mu}\delta e^{\mu}_{a} + \eta^{ba}e^{*c}_{\mu}\delta e^{\mu}_{a}\right) \\ &= \frac{1}{2}\int d^n x\sqrt{g}\, T_{\beta\alpha}\left(e^{\beta}_{c}\eta^{ca}\delta e^{\alpha}_{a} + e^{\alpha}_{b}\eta^{bc}\delta e^{\beta}_{c}\right) \\ &= \frac{1}{2}\int d^n x\sqrt{g}\, T_{\beta\alpha}\, \delta g^{\alpha\beta}\end{aligned}$$

Here, of course, $T_{\beta\alpha} = e^{*c}_{\beta}e^{*b}_{\alpha}T_{cb}$. Thus the frame variation leads to the same energy-momentum tensor as Hilbert's metric variation.

Belinfante Energy-Momentum tensor: In flat space we have global Poincaré symmetry. The Poincaré group is generated by global charges obtained by integrating a local current

$$J^\mu = \frac{1}{2}\theta^{ab} M^\mu{}_{ab} - \epsilon^\nu T^\mu{}_\nu.$$

The skew-symmetric coefficients θ^{ab} parametrize an infinitesimal rotation, and the vector ϵ^ν is an infinitesimal translation. Here

$$T^\mu{}_\nu = \frac{\partial L}{\partial \phi_\mu}\partial_\nu \phi_\nu - \delta^\mu_\nu L$$

is the canonical Noether momentum, and

$$M^\mu{}_{\nu\lambda} = (x_\nu T^\mu{}_\lambda - x_\lambda T^\mu{}_\nu) + S^\mu{}_{\nu\lambda}$$

is the angular-momentum current, with $S^\mu{}_{\nu\lambda}$ being the contribution of the intrinsic (spin) angular momentum. Since the Poincaré group is a global symmetry of the system, this current is locally conserved, $\partial_\mu J^\mu = 0$, for all values of the parameters θ^{ab} and ϵ^ν. In particular we have both

$$\partial_\mu T^{\mu\nu} = 0$$

and

$$\partial_\mu M^\mu{}_{\nu\lambda} = 0.$$

Now

$$\partial_\mu M^\mu{}_{\nu\lambda} = 0 \quad \Leftrightarrow \quad \partial_\mu S^\mu{}_{\nu\lambda} = T_{\lambda\nu} - T_{\nu\lambda}.$$

Thus an exchange of angular momentum between the orbital part and the spin part implies a non-symmetric canonical energy-momentum tensor.

The Belinfante tensor[9]

$$T_B^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda (S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu})$$

is constructed from the above objects so as to be symmetric yet still conserved. The total angular momentum is given by

$$\begin{aligned} M^{\mu\nu} &= \int_{x^0=t} (x^\mu T^{0\nu} - x^\nu T^{0\mu} + S^{0\mu\nu}) d^3x \\ &= \int_{x^0=t} (x^\mu T_B^{0\nu} - x^\nu T_B^{0\mu}) d^3x, \end{aligned}$$

[9] F. J. Belinfante, *On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields*, Physica **7** (1940) 449-474.

where we have integrated by parts in the second line. We see that the Belinfante tensor tacitly provides the spin contribution to the total angular momentum. This shows that the added term in Belinfante tensor is nothing but the "bound momentum" associated with gradients of the intrinsic angular momentum. It is a direct analogue of the $\mathbf{J}_{\text{bound}} = \nabla \times \mathbf{M}$ "bound current" associated with a magnetization density $\mathbf{M}$. The source of the magnetic field is $\mathbf{J}_{\text{tot}} = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{bound}}$, and it is not unreasonable that the that the momentum density induced by a spin gradient should similarly be a source of the gravitational field.

The Belinfante tensor is the Hilbert tensor: Take the action to be $S_{\text{eff}}(\mathbf{e}_a, \boldsymbol{\omega}^{ab})$, but regard $\boldsymbol{\omega}^{ab}$ to be determined by $\mathbf{e}_a$ *via* the condition of being metric compatible and torsion free. Define the "spin current" $S^{\mu}{}_{ab}$ to be

$$S^{\mu}{}_{ab} = \frac{2}{\sqrt{g}} \left(\frac{\delta S_{\text{eff}}}{\delta \omega^{ab}{}_{\mu}} \right)_{\mathbf{e}_a},$$

and also a "canonical" energy momentum tensor $T^{(0)}_{cb}$ by

$$T^{(0)}_{cb} \eta^{ca} e^{*b}_{\mu} = \frac{1}{\sqrt{g}} \left(\frac{\delta S_{\text{eff}}}{\delta e^{\mu}_{a}} \right)_{\boldsymbol{\omega}^{ab}}.$$

Then

$$\delta S_{\text{eff}} = \int d^n x \sqrt{g} \left\{ T^{(0)}_{cb} \eta^{ca} e^{*b}_{\mu} \delta e^{\mu}_{a} + \frac{1}{2} S^{\mu}{}_{ab} \delta \omega^{ab}{}_{\mu} \right\}.$$

Now, for a torsion-free metric connection we have

$$\mathbf{e}_i(\omega^i{}_{jk} - \omega^i{}_{kj}) = \nabla_{\mathbf{e}_k} \mathbf{e}_j - \nabla_{\mathbf{e}_j} \mathbf{e}_k = [\mathbf{e}_k, \mathbf{e}_j] \equiv \mathbf{e}_i f^i{}_{kj},$$

implying

$$\omega_{ijk} = -\frac{1}{2}(f_{ijk} + f_{jki} - f_{kij}).$$

The connection is therefore determined by the Lie brackets of the frames.

We now vary the frame, and compute the change in f_{ijk}. After some index gymnastics, we find

$$\delta f_{ijk} = \eta_{ib} \left\{ (\nabla_j [e^{*b}_{\alpha} \delta e^{\alpha}_{k}] - \nabla_k [e^{*b}_{\alpha} \delta e^{\alpha}_{j}]) - (\omega^b{}_{k\mu} \delta e^{\mu}_{j} - \omega^b{}_{j\mu} \delta e^{\mu}_{k}) \right\}.$$

Here the covariant derivative ∇_j is a *frame-bundle* covariant derivative, and so contains connection terms that act on both the b and k indices in $[e^{*b}_{\alpha} \delta e^{\alpha}_{k}]$.

On adding and using the antisymmetry of $\omega_{ij\mu}$ in its first two indices, we find that

$$\begin{aligned}\delta\omega_{ijk} &= -\frac{1}{2}\left\{(\eta_{ib}(\nabla_j[e^{*b}_\alpha\delta e^\alpha_k]-\nabla_k[e^{*b}_\alpha\delta e^\alpha_j])\right.\\ &\quad +\eta_{jb}(\nabla_k[e^{*b}_\alpha\delta e^\alpha_i]-\nabla_i[e^{*b}_\alpha\delta e^\alpha_k])\\ &\quad \left.-\eta_{kb}(\nabla_i[e^{*b}_\alpha\delta e^\alpha_j]-\nabla_j[e^{*b}_\alpha\delta e^\alpha_i])\right\}+\omega_{ij\mu}\delta e^\mu_k.\end{aligned}$$

The very last term spoils the frame-bundle tensor character of $\delta\omega_{ijk}$. Now

$$\omega_{ijk}=\omega_{ij\mu}e^\mu_k$$

and so

$$\delta\omega_{ijk}=(\delta\omega_{ij\mu})e^\mu_k+\omega_{ij\mu}\delta e^\mu_k.$$

Thus $\delta\omega_{ij\mu}$ *is* a frame-bundle tensor, and

$$\begin{aligned}(\delta\omega_{ij\mu})e^\mu_k &= -\frac{1}{2}\left\{(\eta_{ib}(\nabla_j[e^{*b}_\alpha\delta e^\alpha_k]-\nabla_k[e^{*b}_\alpha\delta e^\alpha_j])\right.\\ &\quad +\eta_{jb}(\nabla_k[e^{*b}_\alpha\delta e^\alpha_i]-\nabla_i[e^{*b}_\alpha\delta e^\alpha_k])\\ &\quad \left.-\eta_{kb}(\nabla_i[e^{*b}_\alpha\delta e^\alpha_j]-\nabla_j[e^{*b}_\alpha\delta e^\alpha_i])\right\}.\end{aligned}$$

The combinations $A^b{}_j=[e^{*b}_\alpha\delta e^\alpha_j]$ are a natural analogue of $g^{-1}\delta g$. Also $\eta_{ib}[e^{*b}_\alpha\delta e^\alpha_k]=\mathbf{e}_i\cdot\delta\mathbf{e}_k$ is a frame-bundle tensor defining the change in the frame. Let us set

$$\delta e_{ij}=\mathbf{e}_i\cdot\delta\mathbf{e}_j=\eta_{ib}[e^{*b}_\alpha\delta e^\alpha_j],$$

then

$$\begin{aligned}(\delta\omega_{ij\mu})e^\mu_k &= -\frac{1}{2}\left\{(\nabla_j\delta e_{ik}-\nabla_k\delta e_{ij})\right.\\ &\quad +(\nabla_k\delta e_{ji}-\nabla_i\delta e_{jk})\\ &\quad \left.-(\nabla_i\delta e_{kj}-\nabla_j\delta e_{ki})\right\}.\end{aligned}$$

If the change in the frame corresponds to a rotation, then δe_{ij} is skew symmetric, and most of the terms cancel, leaving

$$(\delta\omega_{ij\mu})e^\mu_k\overset{\text{rotation}}{=}\nabla_k\delta e_{ij},$$

or

$$\delta\omega_{ij\mu}\overset{\text{rotation}}{=}\nabla_\mu\delta e_{ij}.$$

which is the correct gauge transformation rule.

Now we plug the general variation into δS_{eff}. After an integration by parts, we find that

$$\delta S_{\text{eff}} = \int d^n x \sqrt{g} \left\{ T^{(0)}_{cb} + \frac{1}{2} \nabla_a (S_{bc}{}^a + S_{cb}{}^a - S^a{}_{bc}) \right\} \eta^{cd} e^{*b}_\mu \, \delta e^\mu_d .$$

Thus the Hilbert variation gives exactly the Belinfante tensor. This result is due to Leon Rosenfeld[10].

Example: The Dirac equation. The $(+,-,-,\ldots)$, Minkowski signature Lagrangian for the Dirac field is

$$\int d^d x \sqrt{g} \left\{ \frac{i}{2} \left(\bar{\Psi} \gamma^a e^\mu_a \nabla_\mu \Psi - (\nabla_\mu \bar{\Psi}) e^\mu_a \gamma^a \Psi \right) + m \bar{\Psi} \Psi \right\}$$

where the spinor covariant derivatives acting on spinors are

$$\begin{aligned} \nabla_\mu \Psi &= \left(\frac{\partial}{\partial x^\mu} + \frac{1}{8} [\gamma_b, \gamma_c] \omega^{bc}{}_\mu \right) \Psi, \\ \nabla_\mu \bar{\Psi} &= \bar{\Psi} \left(\frac{\overleftarrow{\partial}}{\partial x^\mu} - \frac{1}{8} [\gamma_b, \gamma_c] \omega^{bc}{}_\mu \right) \\ &\equiv \frac{\partial}{\partial x^\mu} \bar{\Psi} - \frac{1}{8} \bar{\Psi} [\gamma_b, \gamma_c] \omega^{bc}{}_\mu . \end{aligned}$$

The equations of motion are

$$\begin{aligned} i \gamma^\mu \nabla_\mu \Psi + m \Psi &= 0, \\ -i \nabla_\mu \bar{\Psi} \gamma^\mu + m \bar{\Psi} &= 0, \end{aligned}$$

and the canonical energy-momentum tensor and the spin current are, respectively,

$$\begin{aligned} T^{(0)}_{bc} &= \frac{i}{2} \left(\bar{\Psi} \gamma_b (\nabla_c \Psi) - (\nabla_c \bar{\Psi}) \gamma_b \Psi \right), \\ S^a{}_{bc} &= \frac{i}{8} \bar{\Psi} \{ \gamma^a, [\gamma_b, \gamma_c] \} \Psi . \end{aligned}$$

[10] L. Rosenfeld, *Sur le tenseur D'Impulsion-Énergie*, Acad. Roy. Belgique, Mémoirs de la Classes de Sciences **18** (1938), fasicule 6 (1940) 1-30.

(No contribution from $\sqrt{g}$ if we us the equations of motion.) We note that

$$\begin{aligned}\{\gamma_a, [\gamma_b, \gamma_c]\} &= 4\gamma_a\gamma_b\gamma_c, \quad \text{if } a, b, c, \text{ distinct}, \\ &= 0, \qquad\qquad \text{otherwise.}\end{aligned}$$

As a consequence S_{abc} is totally antisymmetric. Now, using this result, and again the equations of motion, we find that

$$\nabla_a S^a{}_{bc} = T^{(0)}_{cb} - T^{(0)}_{bc},$$

and the Belinfante tensor becomes

$$\begin{aligned}T_{bc} &= T^{(0)}_{bc} + \frac{1}{2}(T^{(0)}_{cb} - T^{(0)}_{bc}) \\ &= \frac{1}{2}(T^{(0)}_{bc} + T^{(0)}_{cb}).\end{aligned}$$

The on-shell Belinfante tensor is therefore seen to be the symmetrized canonical tensor.

That $\nabla_a S^a{}_{bc} = T^{(0)}_{cb} - T^{(0)}_{bc}$ is perhaps not obvious. To see how it goes consider the special case

$$\begin{aligned}\nabla_a S^a{}_{12} &= \frac{i}{4}\left((\nabla_0\bar{\Psi})\gamma^0[\gamma_1,\gamma_2]\Psi + (\nabla_3\bar{\Psi})\gamma^3[\gamma_1,\gamma_2]\Psi + \right. \\ &\qquad \left. + \bar{\Psi}[\gamma_1,\gamma_2]\gamma^0\nabla_0\Psi + \bar{\Psi}[\gamma_1,\gamma_2]\gamma^3\nabla_3\Psi\right).\end{aligned}$$

We would like to make use of the equations of motion $\gamma^a\nabla_a\Psi = 0$ (we ignore the masses as their contributions cancel) *etc.*, but the antisymmetry of the spin current means that only two ($a = 0, 3$) of the necessary terms are present in $S^a{}_{12}$. Adding and subtracting the missing ($a = 1, 2$) terms gives

$$\begin{aligned}\nabla_a S^a{}_{12} &= -\frac{i}{4}\left((\nabla_1\bar{\Psi})\gamma^1[\gamma_1,\gamma_2]\Psi + (\nabla_2\bar{\Psi})\gamma^2[\gamma_1,\gamma_2]\Psi \right. \\ &\qquad \left. + \bar{\Psi}[\gamma_1,\gamma_2]\gamma^1\nabla_1\Psi + \bar{\Psi}[\gamma_1,\gamma_2]\gamma^2\nabla_2\Psi\right) \\ &= -\frac{i}{2}\left((\nabla_1\bar{\Psi})\gamma_2\Psi - (\nabla_2\bar{\Psi})\gamma_1\Psi - \bar{\Psi}\gamma_2\nabla_1\Psi + \bar{\Psi}\gamma_1\nabla_2\Psi\right) \\ &= T^{(0)}_{21} - T^{(0)}_{12}.\end{aligned}$$

We used

$$\gamma^1[\gamma_1,\gamma_2] = 2\gamma_2, \quad [\gamma_1,\gamma_2]\gamma^1 = -2\gamma_2, \quad \textit{etc.}$$

Material added in 2018/19

The canonical E&M tensor and Mathisson-Papapetrou equations: Consider the *canonical* energy-momentum tensor

$$T^{(0)a}{}_b = \frac{i}{2}\left(\bar{\Psi}\gamma^a\nabla_b\Psi - (\nabla_b\bar{\Psi})\gamma^a\Psi\right)$$

which is the a-th component of the current of the momentum k_b. Using the equations of motion we find that

$$\nabla_\mu T^{(0)\mu}{}_\nu = \frac{1}{2}S^{\mu ab}R_{ab\mu\nu},$$

where

$$S^{\mu bc} = \frac{i}{8}\bar{\Psi}\{\gamma^\mu[\gamma^b,\gamma^c]\}\Psi$$

is the spin current. At first sight it looks as if the canonical energy-momentum tensor is not conserved in a gravitational field, however the antisymmetry of S^{abc} coupled with the first Bianchi identity makes the spin-gravity coupling vanish.

Here are the details of the derivation:

$$T^{(0)\mu}{}_\nu = \frac{i}{2}\left(\bar{\Psi}\gamma^\mu\nabla_\nu\psi - (\nabla_\nu\bar{\Psi})\gamma^\mu\Psi\right)$$

and (ignoring the masses as their contribution cancels) $\nabla_\mu\bar{\Psi}\gamma^\mu = 0$, $\gamma^\mu\nabla_\mu\Psi = 0$. Thus

$$\begin{aligned}
\nabla_\mu T^{(0)\mu}{}_\nu &= \frac{i}{2}\left(\bar{\Psi}\gamma^\mu(\nabla_\mu\nabla_\nu\Psi) - (\nabla_\mu\nabla_\nu\bar{\Psi})\gamma^\mu\Psi\right)\\
&= \frac{i}{2}\left(\bar{\Psi}\gamma^\mu((\nabla_\nu\nabla_\mu + [\nabla_\mu,\nabla_\nu])\Psi) - ((\nabla_\nu\nabla_\mu + [\nabla_\mu,\nabla_\nu])\bar{\Psi})\gamma^\mu\Psi\right)\\
&= \frac{i}{2}\left(\bar{\Psi}\gamma^\mu[\nabla_\mu,\nabla_\nu]\Psi) - ([\nabla_\mu,\nabla_\nu]\bar{\Psi})\gamma^\mu\Psi\right).
\end{aligned}$$

But

$$[\nabla_\mu,\nabla_\nu]\Psi = \frac{1}{8}[\gamma^a,\gamma^b]\Psi R_{ab\mu\nu}, \quad [\nabla_\mu,\nabla_\nu]\bar{\Psi} = -\frac{1}{8}\bar{\Psi}[\gamma^a,\gamma^b]R_{ab\mu\nu},$$

so

$$\nabla_\mu T^{(0)\mu}{}_\nu = \frac{i}{16}\left(\bar{\Psi}\gamma^\mu[\gamma^a,\gamma^b]\Psi + \bar{\Psi}[\gamma^a,\gamma^b]\gamma^\mu\Psi\right)R_{ab\mu\nu}. = \frac{1}{2}S^{\mu ab}R_{ab\mu\nu}.$$

Euclidean Dirac with torsion: With torsion present, we need to take care in integrating by parts. Recall that

$$\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g} = \frac{1}{2}g^{\alpha\beta}\partial_\mu g_{\alpha\beta}$$

and that the condition for metric compatibility is

$$\partial_\mu g_{\alpha\beta} - \Gamma^\lambda{}_{\alpha\mu}g_{\lambda\beta} - \Gamma^\lambda{}_{\beta\mu}g_{\alpha\lambda} = 0.$$

From this we deduce that for *any* metric-compatible connection we have

$$\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g} = \Gamma^\alpha{}_{\alpha\mu},$$

but must note that the subscripts on the Christoffel symbol are in a specific order.

The usual Euclidean Dirac operator acting on spinors is

$$\not{D} = \gamma^a e_a^\mu \left(\partial_\mu + \tfrac{1}{2}\sigma^{bc}\,\omega_{bc\mu}\right) = \gamma^a D_a$$

where $\mathbf{e}_a = e_a^\mu\partial_\mu$ is an orthonormal frame,

$$\sigma^{ab} = \tfrac{1}{4}[\gamma^a, \gamma^b]$$

are the skew-hermitian spinor generators of $\mathrm{O}(N)$ that obey

$$[\sigma^{ab}, \sigma^{cd}] = \delta^{bc}\sigma^{ad} - \delta^{ac}\sigma^{bd} + \delta^{ad}\sigma^{bc} - \delta^{bd}\sigma^{ac},$$

and

$$D_a \equiv D_{\mathbf{e}_a} \equiv e_a^\mu D_\mu = e_a^\mu\left(\partial_\mu + \tfrac{1}{2}\sigma_{bc}\,\omega^{bc}{}_\mu\right)$$

is the covariant derivative that acts on Dirac spinors u. On conjugate spinors $u^\dagger \equiv \bar{u}$ we have

$$D_\mu\bar{u} = \partial_\mu\bar{u} - \tfrac{1}{2}\bar{u}\sigma_{bc}\,\omega^{bc}{}_\mu \equiv \bar{u}(\overleftarrow{\partial}_\mu - \tfrac{1}{2}\sigma_{bc}\,\omega^{bc}{}_\mu),$$

so that

$$D_\mu(\bar{u}v) = (D_\mu\bar{u})v + \bar{u}(D_\mu v) = \partial_\mu(\bar{u}v)$$

as befits the derivative of the scalar $\bar{u}v$. Then, using $[\sigma^{ab}, \gamma^c] = \delta^{bc}\gamma^a - \delta^{ac}\gamma^b$, we have

$$D_\mu(\bar{u}\gamma^a v) = \partial_\mu(\bar{u}\gamma^a v) + \omega^a{}_{b\mu}(\bar{u}\gamma^b v)$$

as befits the action on the components $V^a = (\bar{u}\gamma^a v)$ of a contravariant frame-bundle vector.

Hermiticity: Using the natural inner product on spinors, we have

$$\langle \Psi_1 | \not{D}\Psi_2 \rangle = \int d^d x \sqrt{g}\, \Psi_1^\dagger (\not{D}\Psi_2).$$

We wish to integrate by parts so as to compute $\not{D}^\dagger$, and this requires us to evaluate $\partial_\mu(\sqrt{g}e_a^\mu)$. Now

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}e_a^\nu) = \partial_\mu e_a^\nu + e_a^\nu \Gamma^\beta{}_{\beta\mu}.$$

We now set $\mu = \nu$ and sum to obtain

$$\begin{aligned}
\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}e_a^\mu) &= \partial_\mu e_a^\mu + e_a^\mu \Gamma^\beta{}_{\beta\mu} \\
&= \partial_\mu e_a^\mu + e_a^\mu \Gamma^\beta{}_{\mu\beta} + e_a^\mu(\Gamma^\beta{}_{\beta\mu} - \Gamma^\beta{}_{\mu\beta}) \\
&= \partial_\mu e_a^\mu + e_a^\beta \Gamma^\mu{}_{\beta\mu} + e_a^\mu T^\beta{}_{\mu\beta}, \quad \text{(Interchanged dummy indices } \beta \leftrightarrow \mu) \\
&= \nabla_\mu e_a^\mu + e_a^\mu T^\beta{}_{\mu\beta} \\
&= e_b^\mu \omega^b{}_{a\mu} + e_a^\mu T^\beta{}_{\mu\beta}. \\
&= e_b^\mu \omega^b{}_{a\mu} + e_a^\mu T^\beta{}_{\mu\beta}.
\end{aligned}$$

The contribution from this derivative combines with a contribution from the necessity of re-ordering γ^a and σ^{bc} *via*

$$[\gamma^a, \sigma^{bc}] = \delta^{ab}\gamma^c - \delta^{ac}\gamma^b$$

to show that $\not{D}$ as it stands is *not* skew-self-adjoint. We do find, however, that

$$\begin{aligned}
&\int d^d x \sqrt{g}\, \Psi_1^\dagger \left\{ \gamma^a e_a^\mu \left(\partial_\mu + \tfrac{1}{2}\sigma^{bc}\,\omega_{bc\mu} + \tfrac{1}{2}T^\alpha{}_{\mu\alpha} \right) \Psi_2 \right\} \\
= \quad &\int d^d x \sqrt{g} \left\{ -\gamma^a e_a^\mu \left(\partial_\mu + \tfrac{1}{2}\sigma^{bc}\,\omega_{bc\mu} + \tfrac{1}{2}T^\alpha{}_{\mu\alpha} \right) \Psi_1 \right\}^\dagger \Psi_2.
\end{aligned}$$

Thus $\langle\Psi_1|(\not{D}+\frac{1}{2}T^b{}_{ab}\gamma^a)\Psi_2\rangle = -\langle(\not{D}+\frac{1}{2}T^b{}_{ab}\gamma^a)\Psi_1|\Psi_2\rangle$, and the formal adjoint is $(\not{D}+\frac{1}{2}T^b{}_{ab}\gamma^a)^\dagger = -(\not{D}+\frac{1}{2}T^b{}_{ab}\gamma^a)$.

Let us write our skew-self-adjoint Dirac operator as $\not{\mathcal{D}} = \not{D}+\frac{1}{2}T^b{}_{ab}\gamma^a$ and take the action of euclidean signature action Lagrangian for the Dirac field to be

$$S = \int d^dx\sqrt{g}\left\{\frac{1}{2}\left(\Psi^\dagger\not{\mathcal{D}}\Psi - (\not{\mathcal{D}}\Psi)^\dagger\Psi\right) + m\Psi^\dagger\Psi\right\}.$$

Note that the $T^a{}_{\mu a}$ cancel between the two terms, so that we could (as do Onkar *et al.*[11] do) have written the action without them. We get the same equation of motion

$$(\not{\mathcal{D}}+m)\Psi \;=\; 0$$

or

$$\gamma^a e^\mu_a(\partial_\mu + \tfrac{1}{2}\sigma^{ab}\omega_{ab\mu} + \tfrac{1}{2}T^c{}_{\mu c})\Psi + m\Psi = 0.$$

with the extra term though, because of the need to integrate by parts. For the equation of motion for $\Psi^\dagger$ can be written

$$\Psi^\dagger(\overleftarrow{\partial}_\mu - \tfrac{1}{2}\sigma^{ab}\omega_{ab\mu} + \tfrac{1}{2}T^d{}_{\mu d})\gamma^c e^\mu_c - m\Psi^\dagger = 0$$

with $\Psi^\dagger\overleftarrow{\partial}_\mu \equiv \partial_\mu\Psi^\dagger$.

Now Noether's theorem tells us to expect the conservation laws

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\Psi^\dagger\gamma^\mu\Psi) = 0, \quad \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\Psi^\dagger\gamma^5\gamma^\mu\Psi) = -2m\Psi^\dagger\gamma^5\Psi$$

to follow from the classical equations of motion. It is interesting that these equations involve the Levi-Civita connection divergence

$$\mathring{\nabla}_\mu J^\mu \equiv \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}J^\mu)$$

rather than the divergence $\nabla_\mu J^\mu$ constructed from the torsionful connection. These two "divergences" are related by

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}J^\mu) + T^\mu{}_{\beta\mu}J^\beta.$$

[11] O. Parrikar, T. L. Hughes, R. G. Leigh, *Torsion, Parity-odd Response and Anomalies in Topological States*, Phys. Rev. D **90**, (2104) 105004, 1-34.

Note that the contorsion-tensor relation

$$\mathring{\Gamma}^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} - \frac{1}{2} g^{\alpha\lambda} \left(T_{\lambda\mu\nu} + T_{\mu\nu\lambda} - T_{\nu\lambda\mu} \right)$$

confirms that

$$\begin{aligned} \mathring{\Gamma}^{\alpha}{}_{\alpha\nu} &= \Gamma^{\alpha}{}_{\alpha\nu} - \frac{1}{2} g^{\alpha\lambda} \left(T_{\lambda\alpha\nu} + T_{\alpha\nu\lambda} - T_{\nu\lambda\alpha} \right) \\ &= \Gamma^{\alpha}{}_{\alpha\nu} - \frac{1}{2} g^{\alpha\lambda} \left(T_{\lambda\alpha\nu} + T_{\lambda\nu\alpha} \right), \quad (g^{\alpha\lambda} = g^{\lambda\alpha},\, T_{\nu\lambda\alpha} = -T_{\nu\alpha\lambda}) \\ &= \Gamma^{\alpha}{}_{\alpha\nu}. \end{aligned}$$

Let's check the vector-current conservation law.

$$\begin{aligned} g^{-\frac{1}{2}} \partial_\mu (g^{\frac{1}{2}} \Psi^\dagger \gamma^a e^\mu_a \Psi) &= e^\mu_a T^c{}_{\mu c} \Psi^\dagger \gamma^a \Psi \\ &\quad + \Psi^\dagger (\overleftarrow{\partial}_\mu - \tfrac{1}{2} \sigma^{bc} \omega_{bc\mu}) \gamma^a e^\mu_a \Psi \\ &\quad + \Psi^\dagger \gamma^a e^\mu_a (\partial_\mu + \tfrac{1}{2} \sigma^{bc} \omega_{bc\mu}) \Psi \end{aligned}$$

We can put half of the $e^\mu_a T^c{}_{\mu c}$ and $\pm m$ in the parentheses in each of the second and third lines to get

$$\begin{aligned} g^{-\frac{1}{2}} \partial_\mu (g^{\frac{1}{2}} \Psi^\dagger \gamma^a e^\mu_a \Psi) &= \Psi^\dagger (\overleftarrow{\partial}_\mu - \tfrac{1}{2} \sigma^{bc} \omega_{bc\mu} + \tfrac{1}{2} T^c{}_{\mu c}) \gamma^a e^\mu_a \Psi \\ &\quad + \Psi^\dagger \gamma^a e^\mu_a (\partial_\mu + \tfrac{1}{2} \sigma^{bc} \omega_{bc\mu} + \tfrac{1}{2} T^c{}_{\mu c}) \Psi, \end{aligned}$$

so the result is zero by the equations of motion. Similarly

$$\begin{aligned} g^{-\frac{1}{2}} \partial_\mu (g^{\frac{1}{2}} \Psi^\dagger \gamma^5 \gamma^a e^\mu_a \Psi) &= \Psi^\dagger (\overleftarrow{\partial}_\mu - \tfrac{1}{2} \sigma^{bc} \omega_{bc\mu} + \tfrac{1}{2} T^c{}_{\mu c}) \gamma^5 \gamma^a e^\mu_a \Psi \\ &\quad + \Psi^\dagger \gamma^5 \gamma^a e^\mu_a (\partial_\mu + \tfrac{1}{2} \sigma^{bc} \omega_{bc\mu} + \tfrac{1}{2} T^c{}_{\mu c}) \Psi, \\ &= -2m \Psi^\dagger \gamma^5 \Psi. \end{aligned}$$

That the conservation laws should always be $\mathring{\nabla}_\mu J^\mu = 0$ makes sense as it is only this form that gives actual conserved quantities.

Now consider eigenvalue problem

$$\not{\mathcal{D}} u_n \equiv \gamma^a e^\mu_a \left(\partial_\mu + \tfrac{1}{2} \sigma_{bc} \omega^{bc}{}_\mu + \tfrac{1}{2} T^\alpha{}_{\mu\alpha} \right) u_n = i \lambda_n u_n$$

where the u_n are a complete set of mutually orthonormal spinor eigenfunctions. On taking the complex conjugate and exploiting the hermiticity of the euclidean gamma matrices, we have

$$\left(\partial_\mu - \tfrac{1}{2} \sigma^T_{bc} \omega^{bc}{}_\mu + \tfrac{1}{2} T^\alpha{}_{\mu\alpha} \right) u^\dagger_n \gamma^a e^\mu_a = -i \lambda_n u^\dagger_n$$

Writing the Grassmann variable Ψ as $\Psi = \sum_n \chi_n u_n$ and $\bar{\Psi} = \sum_n \bar{\chi}_n u_n^\dagger$ we find

$$S = \sum_n (i\lambda_n + m)\bar{\chi}_n \chi_n.$$

Therefore

$$\langle \bar{\Psi}\gamma^5\gamma^a e_a^\mu \Psi\rangle = \sum_n \frac{u_n^\dagger \gamma^5 \gamma^a e_a^\mu u_n}{i\lambda_n + m}, \quad \langle \Psi^\dagger \gamma^5 \Psi\rangle = \sum_n \frac{u_n^\dagger \gamma^5 u_n}{i\lambda_n + m},$$

and

$$\begin{aligned} \frac{1}{\sqrt{g}}\partial_\mu \sqrt{g}\langle \bar{\Psi}\gamma^5\gamma^a e_a^\mu \Psi\rangle &= \sum_n \frac{2i\lambda_n u_n^\dagger \gamma^5 u_n}{i\lambda_n + m} \\ &= 2\sum_n u_n^\dagger \gamma^5 u_n - 2m\langle \bar{\Psi}\gamma^5\Psi\rangle \end{aligned}$$

by the same algebra as in the torsion-free case. The anomaly density therefore remains as

$$Q(x) = 2\sum_n u_n^\dagger(x)\gamma^5 u_n(x).$$

Now Onkar and others rewrite using the Levi-Civita connection $\mathring{\omega}_{ab\mu}$, the contorsion tensor

$$\omega_{bca} = \mathring{\omega}_{bca} - \tfrac{1}{2}(T_{bca} + T_{abc} - T_{abc}),$$

and the identity

$$\frac{1}{8}\gamma^a[\gamma^b, \gamma^c] = \frac{1}{16}\{\gamma^a, [\gamma^b, \gamma^c]\} + \tfrac{1}{4}(\delta^{ab}\gamma^c - \delta^{ac}\gamma^b)$$

in which $\{\gamma^a, [\gamma^b, \gamma^c]\}$ is totally antisymmetric, to find that the difference

$$(\not{D} + \tfrac{1}{2}T^b{}_{ab}\gamma^a) - e_a^\mu \gamma^a(\partial_\mu + \tfrac{1}{2}\sigma_{ab}\mathring{\omega}^{ab}{}_\mu)$$

is given by

$$\begin{aligned} & -\tfrac{1}{16}\gamma^a[\gamma^b, \gamma^c](T_{bca} + T_{cab} - T_{abc}) + \tfrac{1}{2}T_{bab}\gamma^a \\ = \; & -\tfrac{1}{32}\{\gamma^a[\gamma^b, \gamma^c]\}(T_{bca} + T_{cab} - T_{abc}) - \tfrac{1}{8}(\delta^{ab}\gamma^c - \delta^{ac}\gamma^b)(T_{bca} + T_{cab} - T_{abc}) + \tfrac{1}{2}T_{bab}\gamma^a \\ = \; & -\tfrac{1}{32}\{\gamma^a[\gamma^b, \gamma^c]\}T_{abc} - \tfrac{1}{8}\gamma^c(T_{aca} + 0 - T_{aac}) + \tfrac{1}{8}\gamma^b(0 + T_{aab} - T_{aba}) + \tfrac{1}{2}T_{bab}\gamma^a \\ = \; & -\tfrac{1}{32}\{\gamma^a[\gamma^b, \gamma^c]\}T_{abc}. \end{aligned}$$

Thus

$$(\not{D} + \tfrac{1}{2}T^b{}_{ab}\gamma^a) = e^\mu_a\gamma^a(\partial_\mu + \tfrac{1}{2}\sigma_{ab}\mathring{\omega}^{ab}{}_\mu) - \frac{1}{32}\{\gamma^a, [\gamma^b, \gamma^c]\}T_{abc}.$$

Now, with $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \frac{1}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d$ we have $(\gamma^5)^2 = \mathbb{I}$ so

$$\gamma^k\gamma^5 = \frac{1}{3!}\epsilon_{kabc}\gamma^a\gamma^b\gamma^c,$$

and if a, b, c are all different

$$\gamma^a\gamma^b\gamma^c = \epsilon_{kabc}\gamma^k\gamma^5.$$

Thus

$$\not{\mathcal{D}} = (\not{D} + \tfrac{1}{2}T^b{}_{ab}\gamma^a) = \gamma^a(e^\mu_a(\partial_\mu + \tfrac{1}{2}\sigma_{cd}\mathring{\omega}^{cd}{}_\mu) - \frac{1}{8}\epsilon_{abcd}T^{bcd}\gamma^5).$$

We can also define the totally antisymmetric object

$$\frac{1}{3!}H_{abc}\mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c = \mathbf{e}^a \wedge \left(\tfrac{1}{2}T_{abc}\mathbf{e}^b \wedge \mathbf{e}^c\right)$$

Heat Kernel calculation: Consider the flat space $g_{\mu\nu} = \delta_{\mu\nu} = 0$ (so $\mathring{\omega}^{ab}_\mu = 0$) euclidean-signature operator

$$\not{D} = \gamma^a(\partial_a + \gamma_5 A_a)$$

where

$$A_a = -\frac{1}{8}\epsilon_{abcd}T^{bcd}$$

is *real* and all gamma matrices Hermitian. The coordinate system is Cartesian so there is no distinction between greek and roman indices. We have that

$$\not{D}^\dagger = (-\partial_a + \gamma_5 A_a)\gamma_a = -\gamma^a(\partial_a + \gamma_5 A_a)$$

so $\not{D}$ is skew hermitian. This hermiticity property is quite different from that of the usual chiral gauge field. This is because the present A_a is real rather than being i times a hermitian matrix. It is known that ignoring the non-hermiticity the in the chiral gauge field case, and using $\exp\{t\not{D}^2\}$ as a regulator rather than $\exp\{-t\not{D}^\dagger\not{D}\}$ leads to expressions for the *consistent anomaly* rather than the covariant one.

Now a short computation shows that

$$\not{D}^2 = (\partial_a - \gamma_5 A_a)(\partial_a + \gamma_5 A_a) + \frac{1}{4}\gamma_5[\gamma^a, \gamma^b]\left(F_{ab} - 2(A_a\partial_b - A_b\partial_a)\right).$$

In four dimensions we use Fujikawa's technique to work out the t^{-1} term in

$$\begin{aligned}\langle x|\gamma_5 e^{t\not{D}^2}|x\rangle &= \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\left\{\gamma_5 e^{-ikx}e^{t\not{D}^2}e^{ikx}\right\} \\ &= \int \frac{d^4k}{(2\pi)^4}\mathrm{tr}\left\{\gamma_5 e^{-t|\mathbf{k}|^2}\exp\{tF(\mathbf{k},\partial,\mathbf{A})\}\mathbb{I}\right\}\end{aligned}$$

where

$$F(\mathbf{k},\partial,\mathbf{A}) = 2ik_a\partial_a + (\partial_a - \gamma_5 A_a)(\partial_a + \gamma_5 A_a) + \frac{1}{4}\gamma_5[\gamma^a,\gamma^b]\left(F_{ab} - 2(A_a(\partial_b + ik_b) - A_b(\partial_a + ik_a))\right)$$

Now

$$\int \frac{d^4k}{(2\pi)^4}e^{-t|\mathbf{k}|^2} = \frac{1}{16\pi^2}\frac{1}{t^2}.$$

The only term that can give a γ_5 trace at order $1/t$ comes from the first derivative in

$$(\partial_a - \gamma_5 A_a)(\partial_a + \gamma_5 A_a)$$

hitting the last $\gamma_5 A_a$ thus

$$\langle x|\gamma_5 e^{t\not{D}^2}|x\rangle = \mathrm{tr}\{\mathbb{I}\}\frac{1}{16\pi^2}\left(\frac{1}{t}(\partial_a A_a) + \ldots\right)$$

Note that each $t^2 k_a k_b$ integral gives only *one* factor of t so the k integrals in the t^2 term in the expansion of the exponential change the coefficient of the $\gamma^5[\gamma^a,\gamma^b]F_{ab}$ term from the first order.

This term appears in[12] but was discarded in the course of these author's regularization procedure.

Now, with

$$A_a = -\frac{1}{8}\epsilon_{abcd}e^{*b}_{\lambda}e^{*c}_{\mu}e^{*d}_{\nu}T^{\lambda\mu\nu}$$

we see that $\partial_\mu A_\mu$ is proportional to the Nieh-Yan term.

[12] S-K. Hu, B-L. Young, D. W. McKay, *Functional integral and anomalies in theories with S, P, V, and A currents*, Phys. Rev. D 30, (1984) 836-839.