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Toy Model with $\mathbb{Z}_2$ Vacuum Seizing

The index theorem tells us that in the presence of a topologically non-trivial gauge-field configuration (an instanton) the massless euclidean Dirac operator possesses a zero eigenvalue. The associated Matthews-Salam fermion determinant is therefore zero and the fermion path integral vanishes for this gauge-field configuration. An exception occurs when we use the path integral to compute an n-point function containing a contribution from a Dirac propagator whose mode-expansion contains the zero eigenvalue in a denominator. In this case the denominator zero cancels the zero in the determinant to leave a finite result. The physics interpretation of this mathematical phenomenon is that the field operators terminating the propagator line are absorbing particles that were emitted from the instanton *via* spectral flow. This link between zero modes and particle production is the reason why the index density appears in the anomalous conservation law for the chiral current.

The mathematics of the Dirac index theorem is quite intricate so it is useful to explore a toy model which illustrates the *instanton* $\leftrightarrow$ *zero-eigenvalue* $\leftrightarrow$ *particle-creation* mechanism. To this end, consider the one-dimensional Euclidean-time path integral for a harmonic oscillator coupled to a two-component fermion

$$Z = \int d[\overline{\psi}]d[\psi]d[\Phi]\exp\left\{-\int\left(\frac{1}{2}\dot{\Phi}^2+\frac{1}{2}\Omega^2\Phi^2\right)dt+\int\overline{\psi}(\sigma_1\partial_t+i\alpha\sigma_2\Phi)\psi\,dt\right\}.$$

This simple system was originally introduced as test bed for fermion determinant algorithms in lattice gauge theory[1]. The feature that makes the model interesting is that the differential operator

$$\not{D} \stackrel{\text{def}}{=} \sigma_1\partial_t + i\alpha\sigma_2\Phi(t) = \begin{bmatrix} 0 & \partial_t+\alpha\Phi(t) \\ \partial_t-\alpha\Phi(t) & \end{bmatrix}$$

has a normalizable zero mode whenever $\Phi(t)$ has opposite signs at $t=\pm\infty$. For example

$$\eta_0(t) = \begin{bmatrix} N\exp\{\alpha\int_0^t\Phi(\tau)\,d\tau\} \\ 0 \end{bmatrix} = \begin{bmatrix} u_0(t) \\ 0 \end{bmatrix}$$

[1] J. Polonyi, M. Stone, D. Olson, *Microcanonical simulation of a toy model with vacuum seizing*, Nucl. Phys. B251 [FS13] (1985) 333-352.

obeys $\not{D}\eta_0 = 0$ and the normalization constant

$$N = \left(\int_{-\infty}^{\infty} \exp\left\{ 2\alpha \int_0^t \Phi(\tau) d\tau \right\} dt \right)^{-1/2}$$

is finite when $\alpha\Phi(t)$ changes sign from positive to negative as t goes from $-\infty$ to $+\infty$. We can think of this field configuration as an instanton tunneling event in which Φ interpolates between one asymptotic vacuum state and another.

The classical action is invariant under three global symmetries:

i) Vector U(1):

$$\psi \to e^{i\theta}\psi, \quad \overline{\psi} \to \overline{\psi} e^{-i\theta}.$$

Noether's theorem then yields the conserved charge opeartor

$$Q = \overline{\psi}\sigma_1\psi = \psi^\dagger\psi = \psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2.$$

ii) Axial U(1):

$$\psi \to e^{i\phi\sigma_3}\psi, \quad \overline{\psi} \to \overline{\psi} e^{i\phi\sigma_3}.$$

Again Noether gives us a classically conserved charge

$$Q_3 = \overline{\psi}\sigma_1\sigma_3\psi = \psi^\dagger\sigma_3\psi = \psi_1^\dagger\psi_1 - \psi_2^\dagger\psi_2.$$

If unbroken, this symmetry forces the vacuum expectations $\langle\bar{\psi}(1 \pm \sigma_3)\psi\rangle$ to be zero. We will see, however, that instanton effects lead to a non-zero value for these quantities.

iii) Discrete $\mathbb{Z}_2$:

$$\psi \to \sigma_1\psi, \quad \overline{\psi} \to \overline{\psi}\sigma_1, \quad \Phi(t) \to -\Phi(t).$$

This symmetry will be spontaneously broken.

If we consider only the Bose-field path integral and add a source term we have

$$\begin{aligned} Z_B[J] &= \int d[\Phi] \exp\left\{ -\int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}\Omega^2\Phi^2 + J(t)\Phi \right) dt \right\} \\ &= Z_B[0] \exp\left\{ \frac{1}{2} \int_{-\infty}^{\infty} J(t) G_B(t-t') J(t')\, dt dt' \right\}, \end{aligned}$$

where the Green function is

$$G_B(t-t') = (-\partial_t^2 + \Omega^2)^{-1}_{t,t'} = \frac{1}{2|\Omega|} e^{-|\Omega||t-t'|}.$$

We can use the Green function to compute

$$\begin{aligned}\langle \exp\left\{\int_{t_1}^{t_2} \alpha\Phi(t)\,dt\right\}\rangle &= \exp\left\{\frac{\alpha^2}{2}\int_{t_1}^{t_2}\int_{t_1}^{t_2}\frac{1}{2|\Omega|}e^{-|\Omega||t-t'|}dtdt'\right\} \\ &= \exp\left\{\frac{\alpha^2|t_1-t_2|}{2\Omega^2}\left(1-\frac{1}{|\Omega||t_1-t_2|}\left(1-e^{-|\Omega||t_1-t_2|}\right)\right)\right\},\end{aligned}$$

a result which will be useful later.

The fermion part of the action is

$$S[\psi,\overline{\psi},\Phi] = \int \overline{\psi}(\sigma_1\partial_t + i\alpha\sigma_2\Phi(t))\psi\,dt = \int L\,dt.$$

In the Euclidean path integral $\psi(t)$ and $\bar{\psi}(t)$ are unrelated Grassmann variables but when we pass to operator language we must identify $\overline{\psi} = \psi^\dagger\sigma_1$. Therefore $L = \psi^\dagger(\partial_t - \sigma_3\Phi)\psi$ and comparing with usual euclidean action density $L = \psi^\dagger\partial_t\psi + \hat{H}(\psi^\dagger,\psi)$ we identify the second-quantized Hamiltonian operator as

$$\hat{H} = \alpha\Phi(\psi_2^\dagger\psi_2 - \psi_1^\dagger\psi_1).$$

Suppose initially that Φ is a constant. Then if $|0\rangle$ denotes the no-particle state, the constant-Φ Fermi system has four eigenstates

$$\begin{aligned} |0\rangle, &\qquad E = 0, \\ \psi_2^\dagger|0\rangle, &\qquad E = +\alpha\,\Phi, \\ \psi_1^\dagger|0\rangle, &\qquad E = -\alpha\,\Phi, \\ \psi_1^\dagger\psi_2^\dagger|0\rangle, &\qquad E = 0. \end{aligned}$$

When $\alpha\Phi > 0$, the ground state is $|\text{gnd}\rangle = \psi_1^\dagger|0\rangle$.

Again for Φ constant, we have

$$(\sigma_1\partial_t + i\alpha\sigma_2\Phi)^2 = -(-\partial_t^2 + \alpha^2\Phi^2),$$

and so the euclidean-time Fermi propagator is

$$\begin{aligned}\langle\text{gnd}|T\{\psi(t)\bar{\psi}(t')\}|\text{gnd}\rangle &= G_F(t,t') \\ &= (\sigma_1\partial_t + i\alpha\sigma_2\Phi)^{-1}_{t,t'} \\ &= -(\sigma_1\partial_t + i\alpha\sigma_2\Phi)(-\partial_t^2 + \alpha^2\Phi^2)^{-1}_{t,t'} \\ &= -(\sigma_1\partial_t + i\alpha\sigma_2\Phi)\frac{1}{2|\alpha\Phi|}e^{-|\alpha\Phi||t-t'|}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(\sigma_1 \mathrm{sgn}(t-t') - i\sigma_2 \mathrm{sgn}(\alpha\Phi))e^{-|\alpha\Phi||t-t'|} \\
&= \frac{1}{2}\begin{bmatrix} 0 & \mathrm{sgn}(t-t') - \mathrm{sgn}(\alpha\Phi) \\ \mathrm{sgn}(t-t') + \mathrm{sgn}(\alpha\Phi) & 0 \end{bmatrix} e^{-|\alpha\Phi||t-t'|}.
\end{aligned}
$$

If $\alpha\,\Phi > 0$ we read off the the euclidean-time-ordered 2-point functions

$$
\begin{aligned}
\langle \mathrm{gnd}|\mathrm{T}\{\psi_1(\mathrm{t})\bar{\psi}_2(\mathrm{t}')\}|\mathrm{gnd}\rangle &= \langle \mathrm{gnd}|\mathrm{T}\{\psi_1(\mathrm{t})\psi_1^\dagger(\mathrm{t}')\}|\mathrm{gnd}\rangle = -\theta(t'-t)e^{-\alpha\Phi|t-t'|}, \\
\langle \mathrm{gnd}|\mathrm{T}\{\psi_2(\mathrm{t})\bar{\psi}_1(\mathrm{t}')\}|\mathrm{gnd}\rangle &= \langle \mathrm{gnd}|\mathrm{T}\{\psi_2(\mathrm{t})\psi_2^\dagger(\mathrm{t}')\}|\mathrm{gnd}\rangle = \quad\theta(t-t')e^{-\alpha\Phi|t-t'|},
\end{aligned}
$$

which are consistent with the ground state being $|\mathrm{gnd}\rangle = \psi_1^\dagger|0\rangle$. In the operator picture the minus sign in the first expression arises from the need to anticommute $\psi_1(t)$ past $\bar{\psi}_2(t')$ so that the annihilation operator ψ_1 sits next to $|\mathrm{gnd}\rangle = \psi_1^\dagger|0\rangle$. When $\alpha\Phi < 0$ the propagator neatly rearranges itself so as to be consistent with the new ground state $|\mathrm{gnd}\rangle = \psi_2^\dagger|0\rangle$.

Note that there are no diagonal terms in the propagator matrix so that $\langle\bar{\psi}(1 \pm \sigma_3)\psi\rangle$ is zero—as anticipated in the discussion of the axial symmetry in (ii).

We can treat the model for non-constant Φ by exploiting the same strategy that Casher, Kogut and Susskind use for the Schwinger model[2]. We introduce a new field $\phi(t)$ and set

$$
\psi(t) = e^{\sigma_3\phi(t)}\psi_0(t), \quad \overline{\psi}(t) = \overline{\psi}_0(t)e^{\sigma_3\phi(t)},
$$

so that

$$
S = \int \overline{\psi}_0(\sigma_1\partial_t + \sigma_1\sigma_3\partial_t\phi + i\alpha\sigma_2\Phi)\psi_0\, dt.
$$

Now $\sigma_1\sigma_3 = -i\sigma_2$ so when we take $\partial_t\phi = \alpha\Phi(t)$ the integrand becomes independent of Φ. A formal manipulation of the Matthews-Salam functional determinant $\mathrm{Det}\,\not{D}$ then gives

$$
\begin{aligned}
Z_F[\Phi] &\stackrel{\mathrm{def}}{=} \int d[\overline{\psi}]d[\psi]\exp S[\overline{\psi},\psi,\Phi] \\
&= \mathrm{Det}[\sigma_1\partial_t + i\alpha\sigma_2\Phi(t)] \\
&= \mathrm{Det}\left[\exp\{-\sigma_3\phi(t)\}\sigma_1\partial_t\exp\{-\sigma_3\phi(t)\}\right] \\
&= \mathrm{Det}[\exp\{-\sigma_3\phi(t)\}]Z_F[0]\,\mathrm{Det}[\exp\{-\sigma_3\phi(t)\}] \\
&= Z_F[0]\,\mathrm{Det}[\exp\{-\sigma_3\phi(t)\}]^2 \\
&= Z_F[0]\exp\left\{-2\mathrm{Tr}(\sigma_3\phi)\right\}.
\end{aligned}
$$

[2] A Casher, J. Kogut, L. Susskind, *Vacuum polarization and the absence of free quarks* Phys. Rev. D **10**, (1974) 732-745

At the last step we have assumed that we can apply the finite matrix identity $\ln(\det M) = \mathrm{tr}\,(\ln M)$ to a functional determinant $\mathrm{Det}\,(O)$ and functional trace $\mathrm{Tr}\,(O)$ to get

$$\ln(\mathrm{Det}[\exp\{-\sigma_3\phi(t)\}]^2) = \mathrm{Tr}\,[2\ln(\exp\{-\sigma_3\phi(t)\})] = -2\mathrm{Tr}(\sigma_3\phi).$$

Instead of factoring the Matthews-Salam determinant, we can follow Kazuo Fujikawa[3] and interpret the $\exp\{-2\mathrm{Tr}(\sigma_3\phi)\}$ as a Jacobean factor arising from the change in the Grassmann measure

$$d[\overline{\psi}]d[\psi] = d[\overline{\psi}_0 e^{\sigma_3\phi(t)}]d[e^{\sigma_3\phi(t)}\psi_0] = e^{-2\mathrm{Tr}(\sigma_3\phi)}d[\overline{\psi}_0]d[\psi_0].$$

Combined with the invariance of the classical action, this interpretation yields that same expression for $Z_F[\Phi]$ as above. The non-invariance of the Grassmann measure under the chiral transformation means that the classical $\mathrm{U}(1)_A$ symmetry does not survive quantization.

We need some sort of regularization to define the functional determinant and to make sense of the functional trace $\mathrm{Tr}(\sigma_3\phi)$. In next section we will provide this by use of a discrete-time lattice. For the moment we will just state the result: For open boundary conditions we have

$$\exp\{-2\mathrm{Tr}(\sigma_3\phi)\} = \exp\left\{\pm\int_{-\infty}^{\infty}\alpha\Phi(t)\,dt\right\},$$

where the sign is selected by the precise form of the boundary conditions. For periodic and antiperiodic boundary conditions on the interval $[0,T]$ we have

$$\begin{aligned}\exp\{-2\mathrm{Tr}(\sigma_3\phi)\}_{\mathrm{periodic}} &= 4\sinh^2\left\{\frac{1}{2}\int_0^T\alpha\Phi(t)\,dt\right\},\\ \exp\{-2\mathrm{Tr}(\sigma_3\phi)\}_{\mathrm{antiperiodic}} &= 4\cosh^2\left\{\frac{1}{2}\int_0^T\alpha\Phi(t)\,dt\right\}.\end{aligned}$$

Zero temperature quantities are obtained by taking antiperiodic periodic boundary and a large-T limit. This limit selects the largest of the two exponentials in $\cosh^2(\ldots)$ so the effective potential $V(\Phi)$ seen by the Φ field becomes the minimum of the two expressions

$$\frac{1}{2}\Omega^2\Phi^2 \pm \alpha\Phi,$$

[3] K. Fujikawa, *Path-Integral Measure for Gauge-Invariant Fermion Theories*, Phys. Rev. Lett. **42** 1195-1199 (1979); *Path integral for gauge theories with fermions*, Phys. Rev. D **21** 2848-2858 (1980); (E) Phys. Rev. D 22 (1990) 1499.

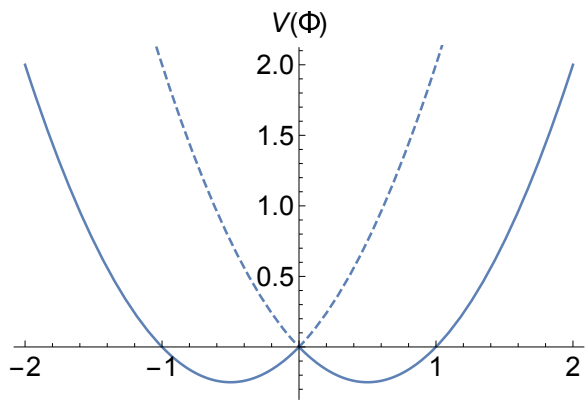

Figure 1: *The effective potential $V(\Phi)$ is the minimum of two shifted harmonic potential wells. The Φ field has a choice of minima, but, the choice having been made, it stays in the selected harmonic potential until the state is acted on by a $\chi_\pm$ operator.*

or

$$V[\Phi] = \mathrm{Min}\left[\frac{1}{2}\Omega^2\left(\Phi \pm \frac{\alpha}{\Omega^2}\right)^2 - \frac{\alpha^2}{2\Omega_2}\right].$$

The $\Phi \leftrightarrow -\Phi$ symmetry is therefore spontaneously broken by Φ sitting at one of the two local minima

$$\langle\Phi\rangle = \mp\frac{\alpha}{\Omega^2}.$$

This symmetry breaking is an example of the Jahn-Teller effect: it costs quadratic $\frac{1}{2}\Omega^2\Phi^2$ spring energy to have a non-zero Φ, but we win back a linear energy from the $-\alpha\Phi$ lowering of the occupied fermion level. For small Φ the linear term always wins and the symmetric value $\Phi = 0$ is unstable. There are therefore two possible ground states

$$\begin{aligned} |0\rangle_+ &= \psi_1^\dagger|0\rangle \otimes |\Phi = +\tfrac{\alpha}{\Omega^2}\rangle, \\ |0\rangle_- &= \psi_2^\dagger|0\rangle \otimes |\Phi = -\tfrac{\alpha}{\Omega^2}\rangle. \end{aligned}$$

Our system is just quantum mechanics: one time dimension and no space dimensions. We would not normally expect spontaneous symmetry breaking in double-well quantum mechanics: instanton tunneling should always always restore the symmetry. Here, however, once we have chosen a vacuum we are

stuck with it. Any attempt by an instanton to tunnel from one well to the other will cause the occupied state to cross from negative to positive energy, and so costs action $e^{-2|\langle\alpha\Phi\rangle|\tau}$ until its effect is undone by an anti-instanton at a time τ later. Instantons must therefore come in closely bound instanton-anti-instanton pairs or else the Fermion determinant will become exponentially small. This suppression of instanton-mediated symmetry restoration is a toy version of the "vacuum seizing" mechanism of Kogut and Susskind[4] that is the source of the photon mass in the Schwinger model. Now consider the operators

$$\begin{aligned}\chi_+ &= \tfrac{1}{2}\bar{\psi}(1+\sigma_3)\psi = \bar{\psi}_1\psi_1 = \psi_2^\dagger\psi_1,\\ \chi_- &= \tfrac{1}{2}\bar{\psi}(1-\sigma_3)\psi = \bar{\psi}_2\psi_2 = \psi_1^\dagger\psi_2.\end{aligned}$$

Under the axial transformations $\psi \to e^{i\phi\sigma_3}\psi$, $\bar{\psi} \to \bar{\psi}e^{i\phi\sigma_3}$ we have

$$\chi_\pm \to e^{\pm 2i\phi}\chi_\pm.$$

This non-invariance would normally preclude these operators from having an expectation value even in the $\mathbb{Z}_2$ symmetry-broken ground state. However, applying χ_+ to $|0\rangle_+$ gives an excited state of energy $2\alpha\Phi$ which can be gobbled up by a Φ-field instanton leaving the system in the other ground state (Figure 2 a). An inverse process (Figure 2b) restores the original ground state. Thus we expect that

$$\begin{aligned}{}_-\langle 0|\chi_+|0\rangle_+ &= A,\\ {}_+\langle 0|\chi_-|0\rangle_- &= A^*,\end{aligned}$$

and

$${}_+\langle 0|\chi_-(t)\chi_+(t')|0\rangle_+ \to |A|^2, \quad |t-t'| \to \infty,$$

where A is a non-zero constant. The $|0\rangle_\pm$ vacua do not satisfy the cluster decomposition property $\langle O(t)O(t')\rangle \to \langle O(t)\rangle\langle O(t')\rangle$ as $|t-t'|$ becomes large because the ${}_+\langle 0|\chi_\pm(t)|0\rangle_+$ expectations are zero. If, however, we define a "theta vacuum"

$$|0\rangle_\theta = \tfrac{1}{\sqrt{2}}\{e^{i\theta}|0\rangle_+ + e^{-i\theta}|0\rangle_-\}$$

and, for example, take $O = \bar{\psi}\psi = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$ we will have

$${}_\theta\langle 0|\bar{\psi}\psi|0\rangle_\theta = \frac{1}{2}(Ae^{+2i\theta} + A^*e^{-2i\theta}),$$

[4] J. Kogut, L. Susskind, *How quark confinement solves the $\eta \to 3\pi$ problem*, Phys. Rev. D **11**, (1975) 3594-3610.

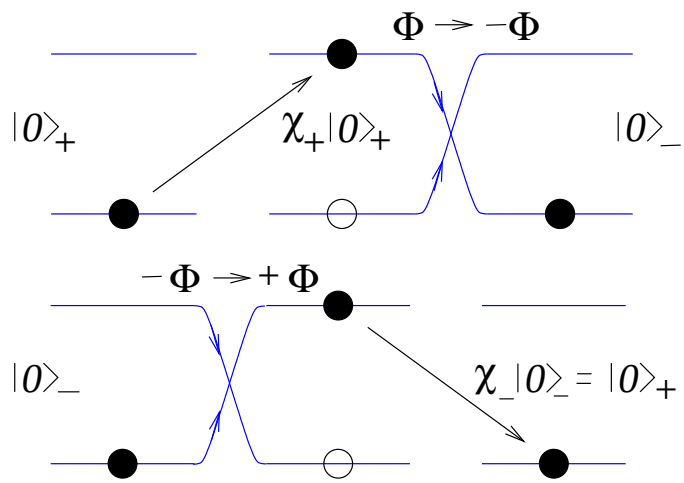

Figure 2: *a) A particle-hole excited state is created by* $\chi_+ = \psi_2^\dagger \psi_1$ *and then devoured by a* $\Phi \to -\Phi$ *instanton. b) An anti-instanton regurgitates the particle-hole excited state, which is then eaten by* $\chi_- = \psi_1^\dagger \psi_2$*.*

and find that the $|0\rangle_\theta$ vacua do satisfy the clustering property.

In the case of open boundary conditions we can arrange for the path integral to automatically compute expectations in the theta vacuum by including a topological term in the action

$$S[\Phi, \psi, \bar{\psi}] \to S[\Phi, \psi, \bar{\psi}] + i\theta\{\mathrm{sgn}[\Phi(-\infty)] - \mathrm{sgn}[\Phi(+\infty)]\}.$$

This term then plays a role analogous to the $i\theta_{QCD}Q$ term in the strong interaction Lagrangian where Q is the integer-valued instanton number.

We can compute the constant A by evaluating the correlation functions in the full interacting theory. If we write $\Phi = \langle\Phi\rangle + \tilde{\Phi}$ the $\tilde{\Phi}$ field is still Gaussian, so we can calculate, for example,

$$\begin{aligned}
{}_+\langle 0|T\{\psi_2(t_1)\bar{\psi}_1(t_2)\}|0\rangle_+ &= {}_+\langle 0|T\{\psi_{2,0}(t_1)\bar{\psi}_{1,0}(t_2)\}|0\rangle_+ \langle e^{\phi(t_2)-\phi(t_1)}\rangle \\
&= \theta(t_1 - t_2)\langle e^{-\int_{t_2}^{t_1} \alpha\Phi(t)dt}\rangle \\
&= \theta(t_1 - t_2)e^{-(\alpha/\Omega^2)(t_1-t_2)}\langle e^{-\int_{t_2}^{t_1} \alpha\tilde{\Phi}(t))dt}\rangle \\
&= \theta(t_1 - t_2)e^{-(\alpha/\Omega^2)(t_1-t_2)}e^{\left\{\frac{\alpha^2}{2\Omega^2}|t_2-t_1| - \frac{\alpha^2}{2\Omega^3}(1-e^{-\Omega|t_1-t_2|})\right\}} \\
&= \theta(t_1 - t_2)e^{\left\{-\frac{\alpha^2}{2\Omega^2}|t_2-t_1| - \frac{\alpha^2}{2\Omega^3}(1-e^{-\Omega|t_1-t_2|})\right\}}
\end{aligned}$$

We see from the asymptotic rate of decay that including the effect of the $\Phi(t)$ fluctuations has halved the energy gap from its static-Φ value of $\alpha\Phi$ to $\alpha\Phi/2$.

Similarly we evaluate the 2-point function of

$$\chi = \bar{\psi}\psi = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 = \psi_2^\dagger\psi_1 + \psi_1^\dagger\psi_2.$$

The $\Phi = 0$ free fermions have

$${}_+\langle 0|T(\chi_0(t_2)\chi_0(t_1))|0\rangle_+ = 1,$$

as for either time order there is a term in the operator expression that takes us from $|0\rangle_+ = \psi_{1,0}^\dagger|0\rangle$ back to $|0\rangle_+$. For the full theory we compute

$$\begin{aligned}
{}_+\langle 0|T(\chi(t_2)\chi(t_1))|0\rangle_+ &= {}_+\langle 0|T(\chi_0(t_2)\chi_0(t_1))|0\rangle_+\langle \exp\left\{2(\phi(t_1)-\phi(t_2))\mathrm{sgn}\,(t_2-t_1)\right\}\rangle \\
&= \left\langle \exp\left\{-2\,\mathrm{sgn}\,(t_2-t_1)\int_{t_1}^{t_2}\alpha\Phi\,dt\right\}\right\rangle \\
&= \exp\left\{-\tfrac{2\alpha^2}{\Omega^2}|t_2-t_1|\right\}\exp\left\{\frac{2\alpha^2}{\Omega^2}|t_2-t_1| - \frac{2\alpha^2}{\Omega^3}\left(1-e^{-\Omega|t_1-t_2|}\right)\right\} \\
&= \exp\left\{-\frac{2\alpha^2}{\Omega^3}\left(1-e^{-\Omega|t_1-t_2|}\right)\right\}.
\end{aligned}$$

For short times

$$\exp\left\{-\frac{2\alpha^2}{\Omega^3}(1-e^{-\Omega|t_1-t_2|})\right\} = \exp\left\{-\frac{2\alpha^2}{\Omega^2}|t_1-t_2| + O(|t_1-t_2|^2)\right\},$$

and this looks like the decay expected for the $E = 2\alpha\langle\Phi\rangle$ excitation we have created by the action of $\bar{\psi}\psi$ on the ground state. As $|t_1-t_2|$ becomes large compared to the Bose field coherence time Ω^{-1}, however, the excitation can be eaten by an instanton and we find

$$\langle\chi(t_1)\chi(t_2)\rangle \to \exp\left\{-\frac{2\alpha^2}{\Omega^3}\right\} = |A|^2.$$

Consequently, although $|0\rangle_+$ is not a clustering state, we can read off that

$$A = \exp\left\{-\frac{\alpha^2}{\Omega^3}\right\}.$$

Repeating the calculation in the theta-vacuum state $|0\rangle_\theta$ will show that that state does possess the cluster decomposition property: $\langle\chi(t_1)\chi(t_2)\rangle \to \langle\chi(t_1)\rangle\langle\chi(t_2)\rangle$ as $|t_1 - t_2|$ becomes large.

The role of the zero mode in the the non-zero expectation for $\chi = \bar{\psi}\psi$ can be seen from the contribution that the $\eta_0(t)$ zero mode makes to the propagator $G_F(t,t')$ in the presence of a $+|\alpha\Phi|$ to $-|\alpha\Phi|$ instanton:

$$G_F(t,t') = \frac{1}{\epsilon}\eta_0(t)\eta_0^\dagger(t') + \ldots = \frac{1}{\epsilon}\begin{bmatrix} u_0(t)u^*(t') & 0 \\ 0 & 0 \end{bmatrix} + \ldots$$

Here ϵ is an eigenvalue which becomes vanishingly small when the instanton is far from any anti-instanton, but cancels against a corresponding small factor ϵ in the fermion determinant. The dots refer to terms that are negligible compared to the zero mode when ϵ is small. Unlike the constant-Φ Green function, this matrix has an entry on the diagonal, and so contributes to $\langle\chi_+(t)\rangle$ when t is close to the instanton where $u_0(t)$ is significant. An anti-instanton has a zero mode that appears on the lower-right diagonal entry in $G_F(t,t')$ and so contributes to $\langle\chi_-\rangle(t)$.

Lattice regularization

If we replace t by x in the Hermitian operator

$$-i\not{D}_t = -i\sigma_1\frac{\partial}{\partial t} + \alpha\sigma_2\Phi(t)$$

that appears in our action and think of it as a *Hamiltonian* H instead of a Lagrangian density, we recognize H as the Hamiltonian of the Jackiw-Rebbi model[5] which famously has an exact zero-energy bound state whenever $\Phi(x)$ changes sign as x goes from $-\infty$ to $+\infty$. The lattice equivalent of this Hamiltonian is the Su-Schrieffer-Heeger (SSH) model[6] for *trans*-polyacetylene which has an exact midgap state when the Peierls dimerization changes parity. This equivalence suggests that we will correctly capture the continuum physics if we use an $x \to t$ version of the SSH model as our lattice regularization.

[5] R. Jackiw, C. Rebbi, *Solitons with fermion number 1/2*, Phys. Rev. D13, (1976) 3398-3409.

[6] W. P. Su, J. R. Schrieffer, A. J. Heeger, *Soliton excitations in polyacetylene*, Phys. Rev. B **22,** (1980) 2099-2111.

We therefore take a one-dimensional temporal lattice with lattice spacing a, and take the Fermi action to be

$$S[\Phi,\bar\psi,\psi]=\sum_n a\frac{1}{2a}\overline{\psi}(n)\left\{\psi(n+1)\exp\{(-1)^n a\Phi(n)\}-\psi(n-1)\exp\{(-1)^{n-1}a\Phi(n)\}\right\}.$$

Here the $\Phi(n)$ variable lives on the link between site n and site $n+1$. We set $\psi(2n-1)\sim\psi_1$, $\psi(2n)\sim\psi_2$ so the formal continuum limit agrees with our continuous-time model.

We write the action as a quadratic form $\frac{1}{2}\bar\psi\not{\mathcal{D}}(\Phi)\psi$:

$$\frac{1}{2}\left(\overline{\psi}(1),\overline{\psi}(2),\overline{\psi}(3),\cdots\right)\begin{pmatrix} 0 & e^{a\Phi(1)} & & & & \\ -e^{a\Phi(1)} & 0 & e^{-a\Phi(2)} & & & \\ 0 & -e^{-a\Phi(2)} & 0 & e^{a\Phi(3)} & & \\ & & -e^{a\Phi(3)} & 0 & e^{-a\Phi(4)} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}\begin{pmatrix}\psi(1)\\ \psi(2)\\ \psi(3)\\ \vdots\end{pmatrix}.$$

Lattice change of variables: Set $a=1$ for convenience and start with

$$S[0,\bar\psi,\psi]=\sum_n\frac{1}{2}\overline{\psi}_0(n)\{\psi_0(n+1)-\psi_0(n-1)\}.$$

Make the change of variables

$$\begin{aligned}\psi_0(n) &= e^{-(-1)^n\phi(n)}\psi(n),\\ \overline{\psi}_0(n) &= \overline{\psi}(n)e^{-(-1)^n\phi(n)},\end{aligned}$$

and find

$$S=\sum_n\frac{1}{2}\overline{\psi}(n)\left\{\psi(n+1)e^{(-1)^n(\phi(n+1)-\phi(n))}-\psi(n-1)e^{(-1)^{(n-1)}(\phi(n)-\phi(n-1))}\right\}$$

We therefore identify $\Phi(n)=(\phi(n+1)-\phi(n))/a$, which is again consistent with the continuum relation $\Phi=\alpha\partial_t\phi$.

In matrix form we can write this as $\not{\mathcal{D}}(\Phi)=A(\phi)\not{\mathcal{D}}(0)A(\phi)$ where

$$A(\phi)=\mathrm{diag}(e^{-\phi(1)},e^{\phi(2)},e^{-\phi(3)}\ldots.e^{\phi(2N)}).$$

Now for an even number of sites with open boundary conditions we have $\det\not{\mathcal{D}}(0)=1$. For an odd number of sites $\det\not{\mathcal{D}}(0)=0$.

For an even number $(2N)$ of sites we therefore have $\det \not{D}(\Phi) = [\det A(\phi)]^2$, or

$$
\begin{aligned}
\det(\not{D}) &\equiv \begin{vmatrix} 0 & e^{a\Phi(1)} & & & & \\ -e^{a\Phi(1)} & 0 & e^{-a\Phi(2)} & & & \\ 0 & -e^{-a\Phi(2)} & 0 & e^{a\Phi(3)} & & \\ & & -e^{a\Phi(3)} & 0 & e^{-a\Phi(4)} & \\ & & & \ddots & \ddots & \ddots \end{vmatrix} \\
&= \exp\{-2(\phi(1) - \phi(2) + \phi(3) - \phi(4) + \cdots + \phi(2N))\} \\
&= \exp\{2a(\Phi(1) + \Phi(3) + \cdots + \Phi(2N-1))\}. \\
&\approx \exp\left\{\int_0^T \Phi(t)\, dt\right\}.
\end{aligned}
$$

The approximation in the last line applies when $\Phi(n)$ varies only slowly from site to site. With $\Phi(n)$ a constant, we recognize that this is the lattice version of $Z_F[\Phi] = \exp\{-TE_0\}$ with $E_0 = -\Phi$. The axial change of variables therefore succeeds in computing the fermion determinant just as it does in the continuum. A finite chain, however, breaks the discrete $\mathbb{Z}_2$ invariance. The open boundary conditions with a strong $(e^{a\Phi})$ bond on the first link have selected the ground state $|0\rangle_+$ and the symmetry breaking in which $\langle\Phi\rangle > 0$. If we want the other ground state, we must place a weak $(e^{-a\Phi})$ bond on the first link.

The reason that the determinant is zero for odd number of sites is that an odd number forces the existence of an instanton and hence an exact zero mode for the Fermi operator[7].

With $2N$ sites and *anti-periodic* boundary conditions, we have

$$
\begin{aligned}
\det \not{D}(\Phi) &= \begin{vmatrix} 0 & e^{a\Phi(1)} & & & & e^{-a\Phi(2N)} \\ -e^{a\Phi(1)} & 0 & e^{-a\Phi(2)} & & & 0 \\ 0 & -e^{-a\Phi(2)} & 0 & e^{a\Phi(3)} & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ -e^{-a\Phi(2N)} & & & & -e^{a\Phi(2N-1)} & 0 \end{vmatrix} \\
&= 2\exp\{a(\Phi(1) - \Phi(2) + \cdots + \Phi(2N-1) - \Phi(2N)\} \\
&\quad + \exp\{-2a(\Phi(2) + \Phi(4) + \cdots + \Phi(2N)\} \\
&\quad + \exp\{+2a(\Phi(1) + \Phi(3) + \cdots + \Phi(2N-1)\}
\end{aligned}
$$

[7] M. Stone, *Zero modes, boundary conditions, and anomalies on the lattice and in the continuum*, Annals of Physics (NY) 155 (1984) 56-84.

$$
\begin{aligned}
&\approx \quad 1+1+\exp\left\{-\int_0^T \Phi(t)\,dt\right\}+\exp\left\{\int_0^T \Phi(t)\,dt\right\}. \\
&= \quad 4\cosh^2\left\{\frac{1}{2}\int_0^T \Phi(t)\,dt\right\}
\end{aligned}
$$

The anti-periodic determinant looks like $\mathrm{Tr}\left(\exp\{-T\hat{H}\}\right)$ with all four fermion states. With these even-number anti-periodic boundary conditions we do not *explicitly* break the $\mathbb{Z}_2$ symmetry and, when T becomes infinite, only the larger of

$$
\exp\left\{\pm\int_0^T \Phi(t)\,dt\right\}
$$

is significant. The effective potential seen by the Φ field is therefore

$$
V(\Phi)=\mathrm{Min}\left[\frac{1}{2}\Omega^2\left(\Phi\pm\frac{\alpha}{\Omega^2}\right)^2-\frac{\alpha^2}{2\Omega_2}\right].
$$

If we select *periodic* boundary conditions by changing the sign of the corner entries in the matrix, the zero-energy states pick up a minus sign. Thus the determinant becomes $\mathrm{Tr}\left((-1)^F\exp\{-TH\}\right)$ with the fermion number F being the number of fermions *added* to the ground state.

Useful Formulae

One-dimensional lattice Laplace Green function:

$$
\int_{-\pi}^{\pi}\frac{dk}{2\pi}\frac{e^{ik(n-n')}}{2(\cosh m-\cos k)}=\frac{1}{2\sinh|m|}e^{-|m||n-n'|}.
$$

The "square" integral:

$$
\frac{1}{2}\int_0^T\int_0^T \exp\{-\Omega|t-t'|\}\,dtdt'=\frac{T}{\Omega}\left(1-\frac{1}{\Omega T}(1-\exp\{-\Omega T\})\right).
$$

Determinants: A $2N$-by-$2N$ determinant of a Jacobi matrix:

$$
\begin{vmatrix}
0 & a_1 & & & \\
b_1 & 0 & a_2 & & \\
0 & b_2 & 0 & a_3 & \\
 & & \ddots & \ddots & \ddots
\end{vmatrix}=(-1)^N(a_1b_1)(a_3b_3)\cdots(a_{2N-1}b_{2N-1}).
$$

The determinant is zero when the number of rows and columns is odd.

A $2N$-by-$2N$ <u>anti</u>-periodic boundary condition determinant:

$$\begin{aligned}
&\begin{vmatrix} 0 & a_1 & & & & b_{2N} \\ -b_1 & 0 & a_2 & & & 0 \\ 0 & -b_2 & 0 & a_3 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ -a_{2N} & & & & -b_{2N-1} & 0 \end{vmatrix} \\
= \quad & \prod_{i=1}^{2N} a_i \prod_{i=1}^{2N} b_i + [(a_2b_2)(a_4b_4)\cdots(a_{2N}b_{2N}) + (a_1b_1)(a_3b_3)\cdots(a_{2N-1}b_{2N-1})] \\
= \quad & (a_1a_3\ldots a_{2N-1} + b_2b_4\ldots b_{2N})(b_1b_3\ldots b_{2N-1} + a_2a_4\ldots a_{2N}).
\end{aligned}$$

Special case $b_i = a_i$ is

$$\begin{vmatrix} 0 & a_1 & & & & a_{2N} \\ -a_1 & 0 & a_2 & & & 0 \\ 0 & -a_2 & 0 & a_3 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ -a_{2N} & & & & -a_{2N-1} & 0 \end{vmatrix} = (a_1a_3\ldots a_{2N-1} + a_2a_4\ldots a_{2N})^2.$$

A $(2N+1)$-by-$(2N+1)$ periodic boundary condition determinant

$$\begin{vmatrix} 0 & a_1 & & & & b_{2N+1} \\ b_1 & 0 & a_2 & & & 0 \\ 0 & b_2 & 0 & a_3 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ a_{2N+1} & & & & b_{2N} & 0 \end{vmatrix} = \prod_{i=1}^{2N+1} a_i + \prod_{i=1}^{2N+1} b_i$$